The complex numbers are _____.

A) an ordered field

✓ B) a field that can't be ordered

C) not a field

---

ordered pair: $(a,b) \neq (b,a)$

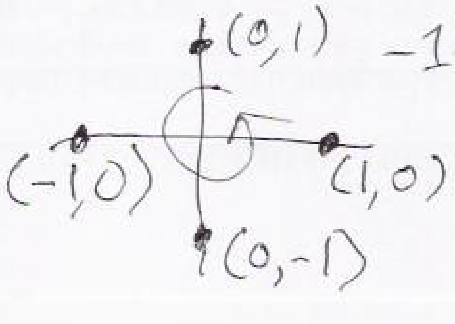
unordered pair: $\{a,b\} = \{b,a\}$

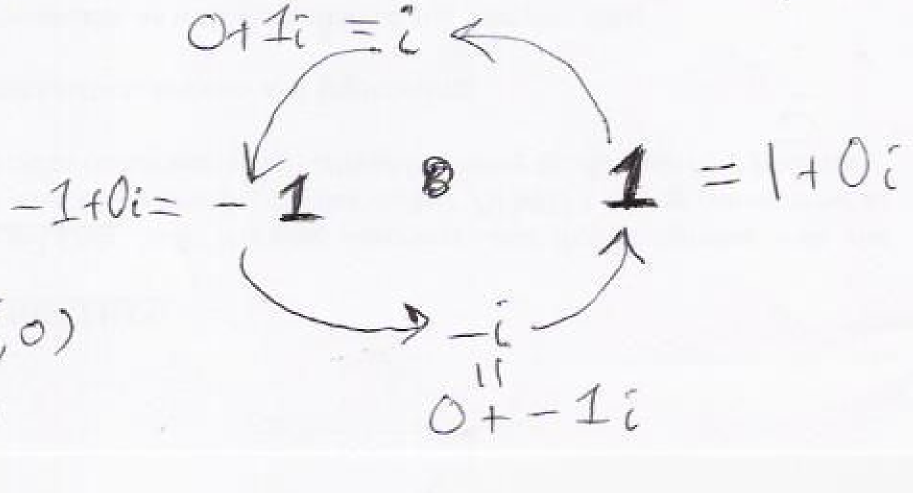
To be an ordered field, you need

- $\forall y,z\ (y<z \Rightarrow \forall x\ \ x+y<x+z)$
- $\forall x,y\ \ (x,y>0 \Rightarrow xy>0)$

$i^2 = -1 \qquad i^3 = i^2 i = -i \qquad i^4 = (i^2)^2 = 1$

$i^5 = i^4 i = i$





$$
\begin{array}{rr}
0 < & x \\
+(-x) & +(-x) \\
\hline
-x < & 0
\end{array}
$$

$$
\begin{array}{rr}
x < & 0 \\
+(-x) & +(-x) \\
\hline
0 < & -x
\end{array}
$$

Aside: proof that $1 > 0$:

If $1 < 0$, then $1 + (-1) < 0 + (-1)$, so $0 < -1$. But then $0 < (-1)(-1) = 1$.

Thus $1 < 0 \Rightarrow 0 < 1$. That is, $1 < 0$ refutes itself. So, $1 < 0$ is false. Therefore, $0 < 1$.

$$i > 0 \Rightarrow i \cdot i > 0$$

$$\Rightarrow \underbrace{i \cdot (i \cdot i)}_{-i} > 0$$

$$\Rightarrow i \cdot (i \cdot (i \cdot i)) > 0$$

$$\Rightarrow i, -1, -i, 1 > 0$$

But $1 > 0 \Rightarrow -1 < 0$

$$i < 0 \Rightarrow -i > 0$$

$$\Rightarrow \underbrace{(-i)(-i)}_{-1} > 0$$

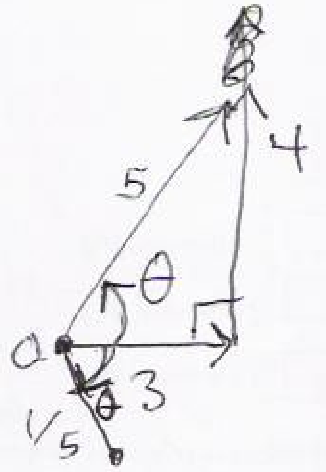
$$\Rightarrow \underbrace{(-i)^3}_{-(i^3) = i} > 0$$

$$\Rightarrow \underbrace{(-i)^4}_{1} > 0$$

$$\Rightarrow -i, -1, i, 1 > 0$$

But $1 > 0 \Rightarrow -1 < 0$

$$\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$$



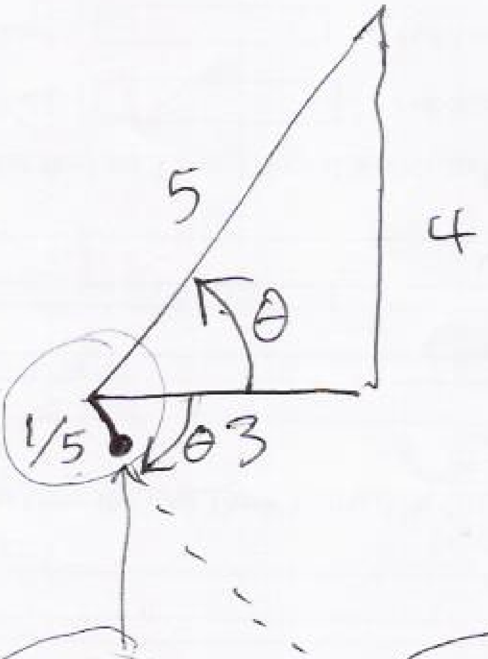
$$9-12i+12i-16i^2$$

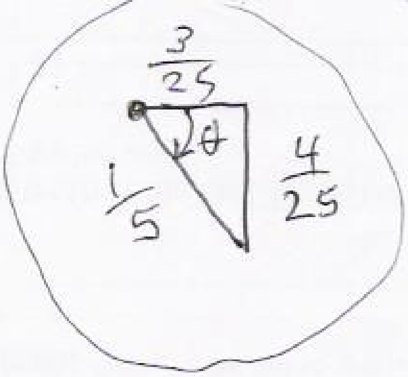
$$9+16=25$$

$$(a+b)(a-b)=a^2-b^2$$

$$(a+bi)(a-bi)=a^2-(bi)^2$$

$$=a^2+b^2$$





$$\sqrt{\left(\frac{3}{25}\right)^2+\left(\frac{4}{25}\right)^2} = \frac{1}{5}$$

Exercise

Prove that the conjugate map

$z \mapsto \bar{z}$ $\quad (a+bi \mapsto a-bi)$

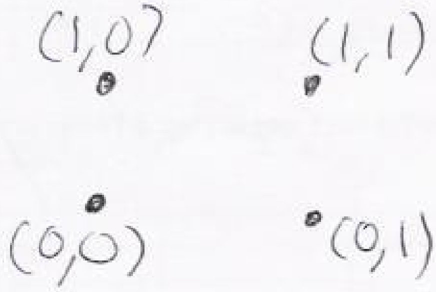
is an isomorphism from

$(\mathbb{C}, +, \cdot)$ to $(\mathbb{C}, +, \cdot)$.

In $\mathbb{R}^{100}$, what is distance from one corner of a $1 \times 1 \times 1 \times 1 \times \cdots \times 1$ cube to the opposite corner?

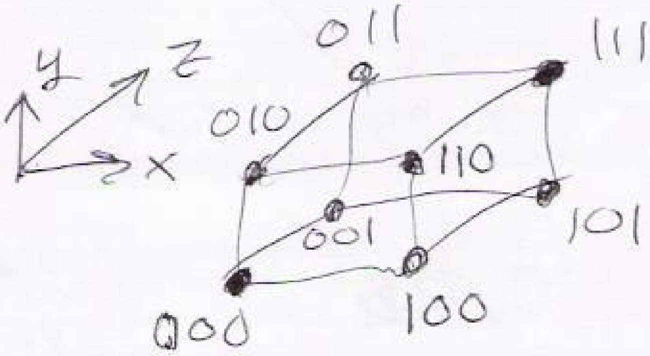
2D cube (square)

$\{0,1\}^2$



3D cube

$\{0,1\}^3$



100D cube $\quad \{0,1\}^{100}$

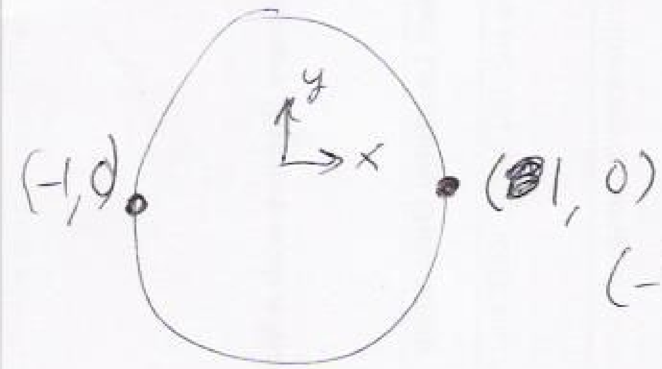
$$|\vec{0} - \vec{1}| = \sqrt{\underbrace{(0-1)^2 + \cdots + (0-1)^2}_{100}}$$

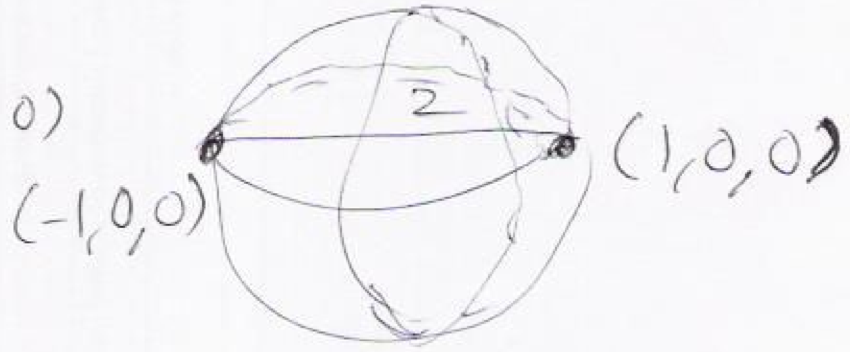
$$\underbrace{(0,\dots,0)}_{100} \qquad \underbrace{(1,\dots,1)}_{100}$$

$$10 = \sqrt{100} = \sqrt{\underbrace{1+\cdots+1}_{100}}$$

$\underbrace{\text{unit}}_{1}$ sphere in $\mathbb{R}^3$: $\{(x,y,z) : x^2+y^2+z^2=1\}$

in $\mathbb{R}^{100}$: $\{(x_1,\dots,x_{100}) : \sum_{n=1}^{100} x_n^2 = 1\}$

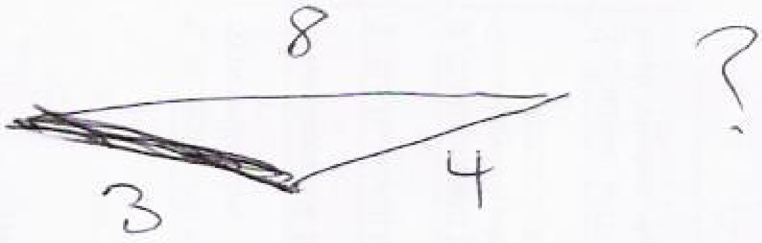


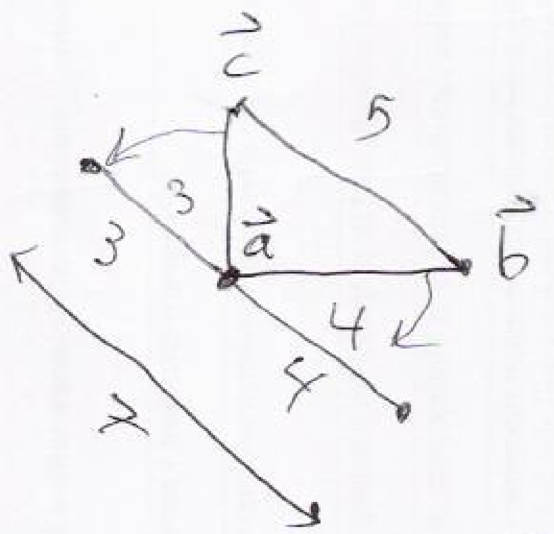


$$|(1,\underbrace{0,\dots,0}_{99}) - (-1,\underbrace{0,\dots,0}_{99})| =$$

$$= \sqrt{(1-(-1))^2 + \underbrace{(0-0)^2 + \cdots + (0-0)^2}_{99}}$$

$$= \sqrt{4} = 2$$





A triangle inequality:

$$|\vec{b}-\vec{c}| \le |\vec{a}-\vec{b}| + |\vec{a}-\vec{c}|$$

$$|\vec{b}-\vec{c}| \le |\vec{b}-\vec{a}| + |\vec{a}-\vec{c}|$$

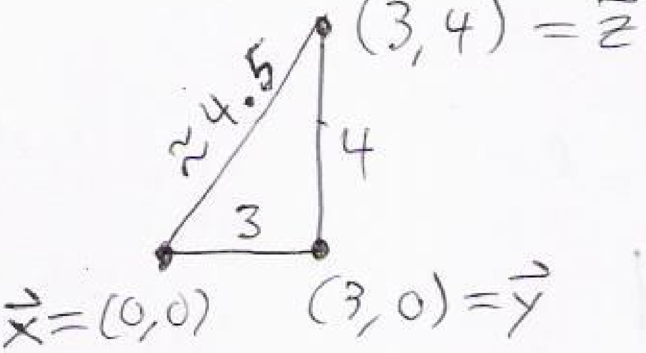
1.37(f)

Exercise: prove or disprove:

If $|\vec{x}|_3 = \sqrt[3]{|x_1|^3+|x_2|^3}$ for all $\vec{x} \in \mathbb{R}^2$, is it true that $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$

$$|\vec{x}-\vec{z}|_3 \le |\vec{x}-\vec{y}|_3 + |\vec{y}-\vec{z}|_3.$$

In the literature, $|\bullet|_3$ is the "L^3-norm."



$$|\vec{x}-\vec{y}|_3 = \overset{\overset{3}{\shortparallel}}{\sqrt[3]{|0-3|^3+|0-0|^3}}$$

$$|\vec{y}-\vec{z}|_3 = \underset{\underset{4}{\shortparallel}}{\sqrt[3]{|3-3|^3+|0-4|^3}}$$

$$|\vec{x}-\vec{z}|_3 = \sqrt[3]{|0-3|^3+|0-4|^3} = \sqrt[3]{27+64}$$

$$= \underbrace{\sqrt[3]{91}}_{\approx 4.5} < \sqrt[3]{125} = 5$$