$x_1 = 0.5742138856\ldots$

$x_2 = 72.1657295433\ldots$

$x_3 = -112.5422200091\ldots$

$x_4 = -0.7613456689\ldots$

$\vdots$

$y = 0.6466\ldots \neq x_1, x_2, x_3, x_4, \ldots$

$0, 1, 2, 3, 4, 5 \longrightarrow 6$

$6, 7, 8, 9 \longrightarrow 4$

$J = \{1, 2, 3, 4, \ldots\}$ $\qquad$ $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

1	0
2	1
3	−1
4	2
5	−2
6	3
7	−3

$\vdots \qquad\qquad \vdots$

$2n$ ——— n

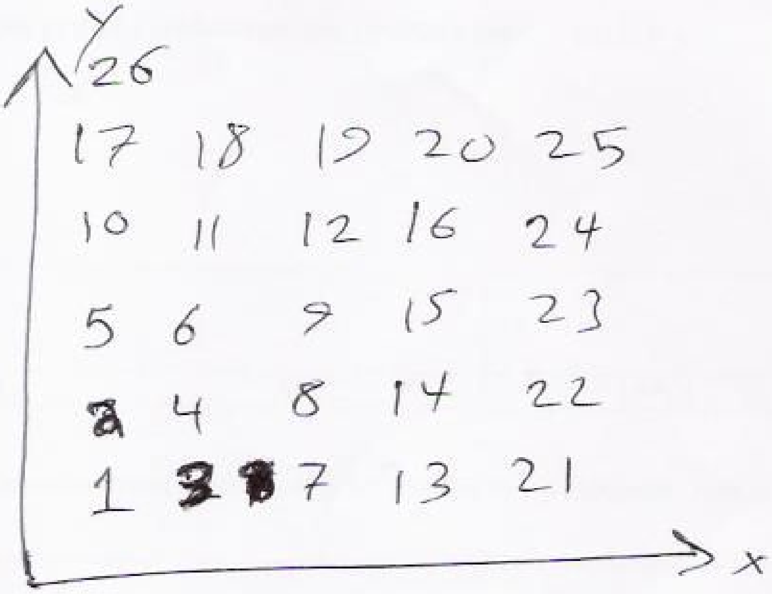
$2n+1$ ——— $-n$

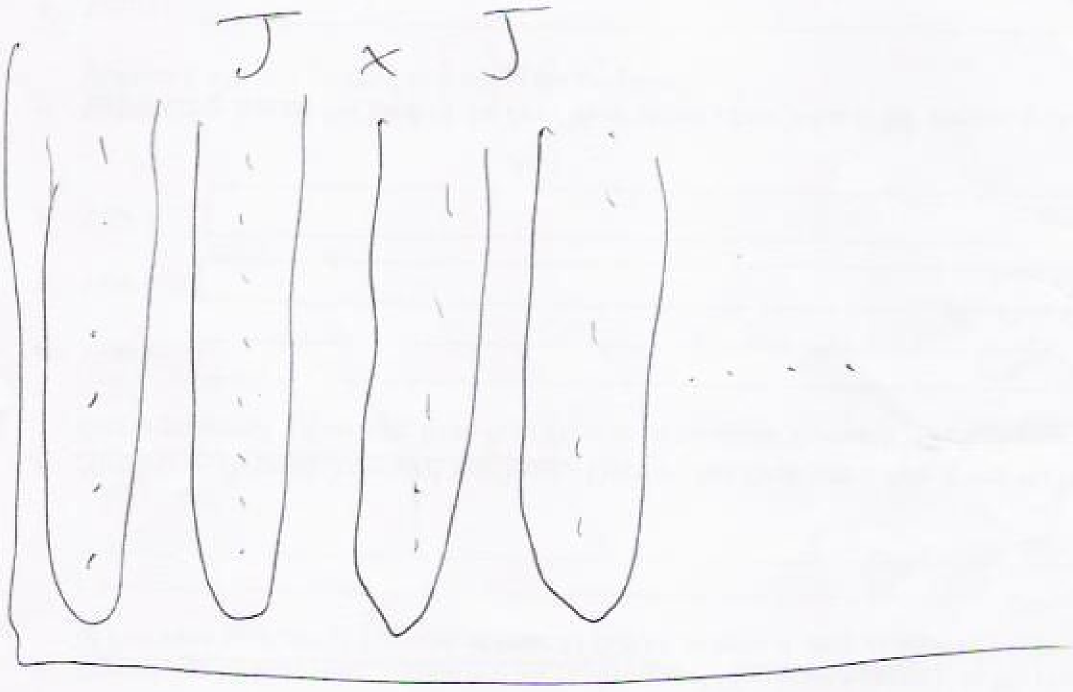
$\vdots \qquad\qquad \vdots$

$J \quad \& \quad \underbrace{J \times J}$

$\{(m,n) : m,n \in J\}$

1	(1, 1)
2	(1, 2)
3	(2, 1)
4	(2, 2)
5	(1, 3)
6	(2, 3)
7	(3, 3)
8	(3, 1)
9	(3, 2)
10	(1, 4)





countable unions of
countable sets are countable

$J \times J \times J \approx J$

$(3, m, n)$

$(2, m, n)$

$(1, m, n)$

$\forall n \quad J^n \approx J$

$J^1 \cup J^2 \cup J^3 \cup J^4 \cup \cdots \approx J$

$\mathbb{Q}^1 \cup \mathbb{Q}^2 \cup \mathbb{Q}^3 \cup \mathbb{Q}^4 \cup \cdots \approx J$

Deal with the devil

1, 3, 5, 7, 9, 11, ...

2, 3, 4, 5, 7, 9, 11, ...

3, 4, 5, 6, 7, 8, 9, 11, ...

4, 5, 6, 7, 8, 9, 10, 11, 12, ...

5, 6, ..., 16, 17, 19, 21, ...

6, ..., 20, 21, 23, 25, ...

⋮

2 for 1, until there are none.

Prove that the set of all (1-variable, x) polynomials with integer coefficients is countable.

$5x^3 - 7$, $-x^5 + 8$, 2,

$x^2 - x + 1$, ...

Use that if $E = \bigcup_{n=1}^{\infty} E_n$ & each E_n is countable, then E is countable

P = set of all polynomials with integer coefficients

$P_n = \{ p \in P : \text{degree of } p \text{ is } n-1 \}$

$P_1 = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$

$P_2 = \{ 5 + 7x, 2 - 3x, -x, x, -7 + x, 14 - 5x, \ldots \}$

$P_2 = \{ m + nx : m, n \text{ integers} \ \& \ n \neq 0 \}$

$P_3 = \{ a + bx + cx^2 : a, b, c \text{ integers} \ \& \ c \neq 0 \}$

$P_4 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, \dots, a_3 \text{ integers} \ \& \ a_3 \neq 0 \}$

$P_n = \{ a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} :$

$a_{n-1} \neq 0 \ \& \ a_0, \dots, a_{n-1} \text{ integers} \}$

$$P = \bigcup_{n=1}^{\infty} P_n$$

If each P_n is countable, then so is P.

Slight simplification:

$P_n' = \{ p \in P : \text{degree}(p) \leq n \}$

$$P = \bigcup_{n=1}^{\infty} P_n'.$$

$P_n' = \{ a_0 + \cdots + a_{n-1} x^{n-1} : a_0, \dots, a_{n-1} \text{ integers} \}$

$P'_1 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ✓

Base case $n=1$ done already:

$$\begin{cases} 2n \longmapsto n \\ 2n+1 \longmapsto -n \\ 1 \longmapsto 0 \end{cases} \quad \text{for all } n = 1, 2, 3, \ldots$$

Assume P'_k is countable;
prove P'_{k+1} is countable.

$$P'_{k+1} = \{a_0 + a_1 x + \cdots a_{k-1} x^{k-1} + a_k x^k : a_0, \ldots, a_k \text{ integers}\}$$

$$P'_{k+1} = \bigcup_{m=1}^{\infty} Q_m \quad \text{where}$$

$$Q_{2m} = \{a_0 + \cdots + a_{k-1} x^{k-1} + m x^k : a_0, \ldots, a_{k-1} \text{ integers}\}$$

$$Q_{2m+1} = \{a_0 + \cdots + a_{k-1} x^{k-1} - m x^k : a_0, \ldots, a_{k-1} \text{ integers}\}$$

$Q_1 = \{a_0 + \cdots + a_{k-1}x^{k-1} : a_0, \ldots, a_{k-1} \text{ integers}\}$

$Q_1 = P_k'$, so Q_1 is countable.

$a_0 + \cdots + a_{k-1}x^{k-1} \mapsto a_0 + \cdots + a_{k-1}x^{k-1} + mx^k$

defines a bijection $Q_1 \rightarrow Q_{2m}$.

Likewise $\exists$ bijection $Q_1 \rightarrow Q_{2m+1}$.

So, every Q_j for $j = 1, 2, 3, \ldots$ is countable.

So, $P_{k+1}' = \bigcup_{m=1}^{\infty} Q_m$ is countable.

By induction, every P_n' is countable. So, $P = \bigcup_{n=1}^{\infty} P_n'$

is countable.