- Every finite subset of $\mathbb{R}^k$ is compact
- $\forall X \subset \mathbb{R}^k$ (X compact $\Longleftrightarrow$ X closed + bounded)

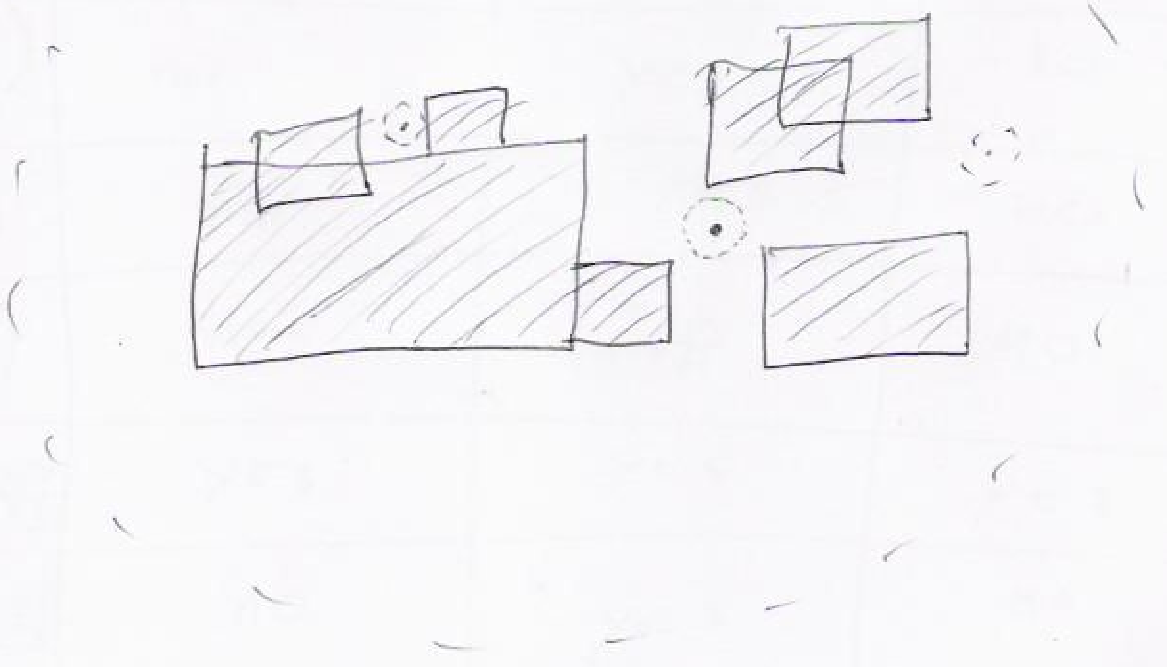
$[0,1]$ closed & bounded in $\mathbb{R}$

$[a,b]$ closed & bounded in $\mathbb{R}$

$[a_1,b_1] \cup \dots \cup [a_n,b_n]$ closed & bounded in $\mathbb{R}$.

A finite union of $\underbrace{\text{closed solid rectangles}}_{\text{2-cell}}$

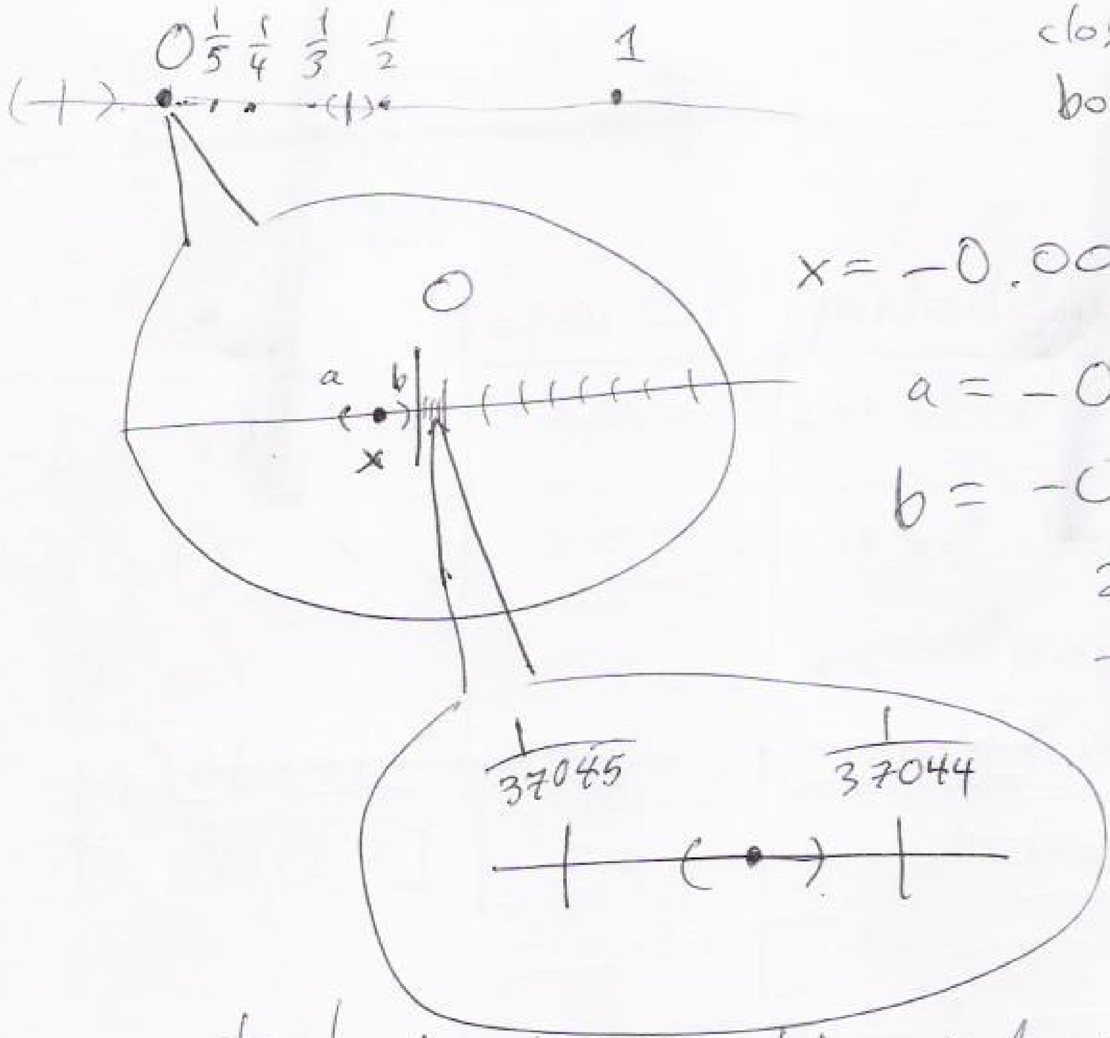
is closed in $\mathbb{R}^2$ & bounded in $\mathbb{R}^2$.



$\{0\} \cup \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ compact $\subset \mathbb{R}$

closed ✓

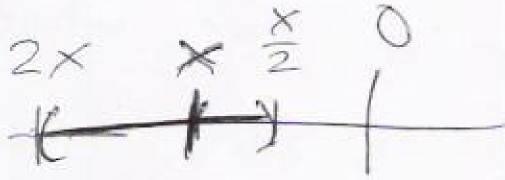
bounded ✓



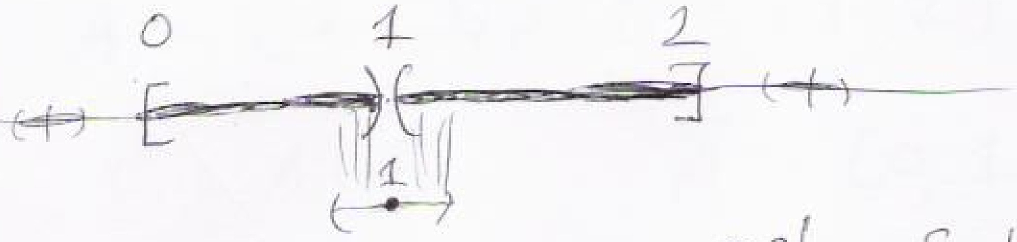
$x = -0.00000342\ldots$

$a = -0.000004$

$b = -0.000002$

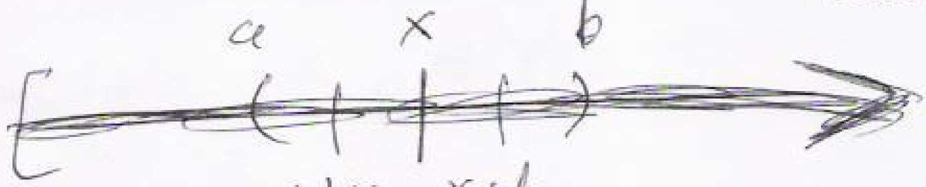


set	closed $\subset \mathbb{R}$?	bounded $\subset \mathbb{R}$?	compact?
$[3, 5)$	no	yes	no
$(0, 1)$	no	yes	no
$[0, \infty)$	yes	no	no
$\{2\} \cup [3, 4]$	yes	yes	yes
$[0, 1) \cup (1, 2]$	no	yes	no



set	open $\subset \mathbb{R}$?	perfect $\subset \mathbb{R}$?	connected?
$[3,5)$	no	no: not closed	yes
$(0,1)$	yes	no: not closed	yes
$[0,\infty)$	no	yes	yes
$\{2\}\cup[3,4]$	no	no: 2 iso.	no
$[0,1)\cup(1,2]$	no	no: not closed	no

perfect = closed without ~~isolated~~ isolated points



$$(\text{connected} \subset \mathbb{R}) \iff (\text{convex} \subset \mathbb{R})$$

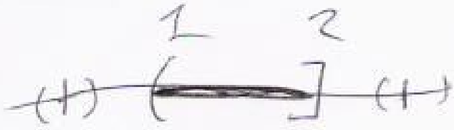
$$(\text{connected} \subset \mathbb{R}^2) \not\Rightarrow (\text{convex} \subset \mathbb{R}^2)$$

 hollow circle

$E = [0,1) \cup (1,2]$

$A = [0,1) \neq \phi \quad B = (1,2] = \phi$

$E = A \cup B \qquad \bar{A} = [0,1]$

$\bar{B} = [1,2]$ 

$\bar{A} \cap B = [0,1] \cap (1,2] = \phi$

$A \cap \bar{B} = [0,1) \cap [1,2] = \phi$

A & B are a separation of E (in $\mathbb{R}$)
So, E is disconnected.

If $Y \subset X$, $\bar{Y} = \{p \in X : p \in Y$

or $\underbrace{p \text{ is a limit point of } Y}_{p \in Y'}\}$

$Y' = \{p \in X : \forall r > 0 \ \exists q \in Y \ 0 < d(p,q) < r\}$

$Y = [0,1] \cap \mathbb{Q}$ $\quad X = \mathbb{Q}$

In X, $\quad Y' = Y$.

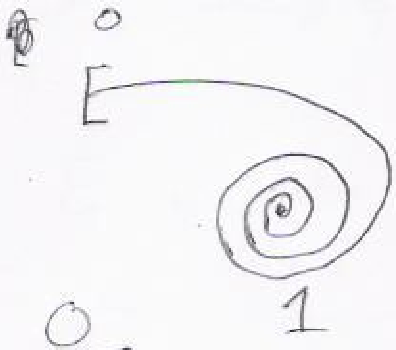
In $\mathbb{R}$, $\quad Y' = [0,1]$

Some heuristics / intuitions:

<u>disconnected</u>: pull it into two distant pieces without tearing

<u>compact</u>: no holes or all holes filled in.

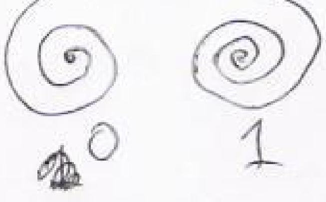
not compact



$[0,1)$ not compact

$[0,\infty)$ not compact

$(0,1)$ not compact



$[0,1] \cap \mathbb{Q}$ not compact:

hole @ every irrational $x \in [0,1]$

$E \subset \mathbb{R}$, E compact & countable:

$E = \{0\} \cup \{1/n : n = 1, 2, 3, \dots\}$

$0 \longleftrightarrow 1$

$1/1 \longleftrightarrow 2$

$1/2 \longleftrightarrow 3$

$1/3 \longleftrightarrow 4$

$\vdots$

E is countable

$\forall P$ perfect $\subset \mathbb{R}^k$

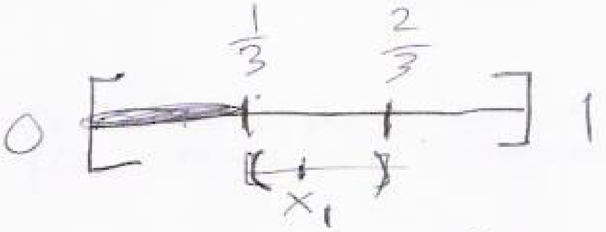
$P = \emptyset$ or P uncountable.

To prove this, you need the fact if

$K_1 \supset K_2 \supset K_3 \supset \cdots$

all nonempty, then

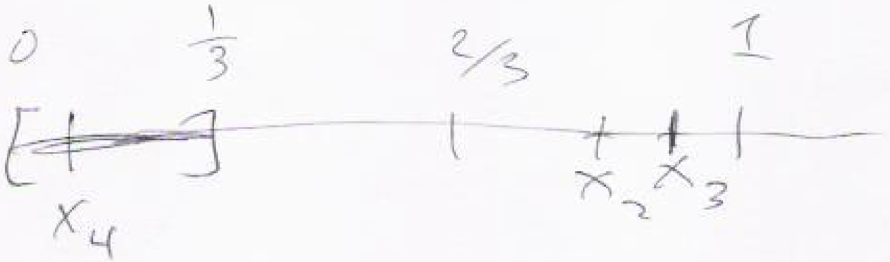
$\bigcap_{n=1}^{\infty} K_n$ nonempty.

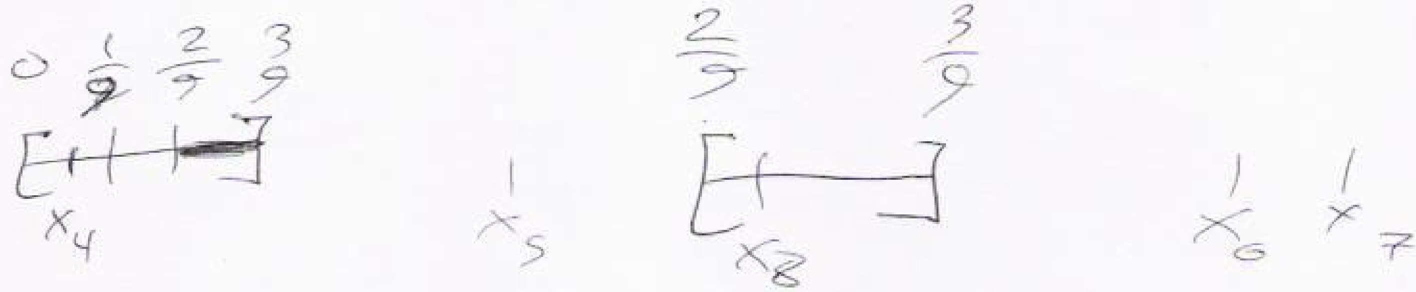


$[0,1]$ is uncountable.

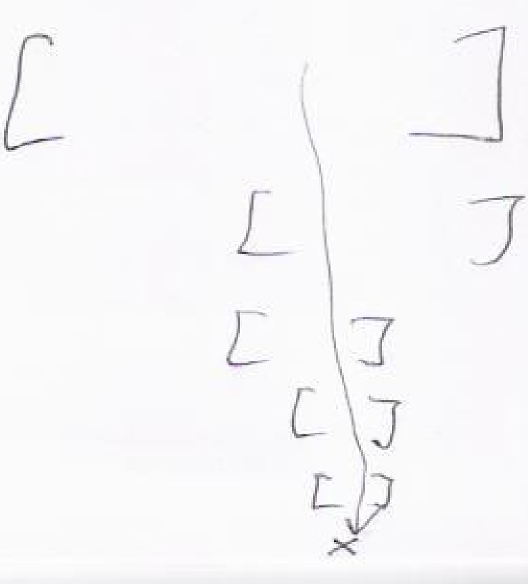
To prove this, suppose $\forall n \; x_n \in [0,1]$.

We need to find $x \in [0,1]$ such that $x \neq x_1, x_2, x_3, \ldots$





Keep picking smaller subintervals to avoid all the x_n's



You need compactness:

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset.$$