Reminders (Sept. 19)

- Do ~2 pages of works towards solving exercises & visit my office for ~15 minutes — counts as 1 extra quiz.
- 9/26: review; 10/1: test
- Choose ultralimits ~~of~~ higher differences tonight.
- Bring USB drive or laptop to my office if you need help installing TeX Live.

4335 Quiz 9/19

In a compact metric space, every sequence ___________.

A) converges
B) has a convergent subsequence ✓
C) diverges
D) has a divergent subsequence

$X = [0, 1] \quad d(x,y) = |x-y|$

→ compact metric space

$0, 1, 0, 1, 0, 1, 0, 1, \ldots$ diverges

$0, 0, 0, 0, 0, 0, 0, 0, \ldots$ converges

→ all subsequences converge

Proof that every sequence $\{p_n\}_{n=1}^{\infty}$ in $[0,1]$ has a convergent subsequence:

There are infinitely many $n \in \{1, 2, 3, \ldots\}$.

Either $p_n \in [0, \frac{1}{2}]$ for ∞-many n,
or $p_n \in [\frac{1}{2}, 1]$ for ∞-many n,
or both.

Let $E_0 = \{1, 2, 3, \ldots\}$

$I_0 = [0, 1]$.

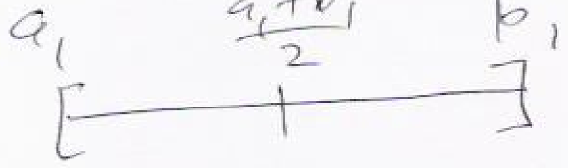
$I_1 = [0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$

and $E_1 = \{n : p_n \in I_1\}$

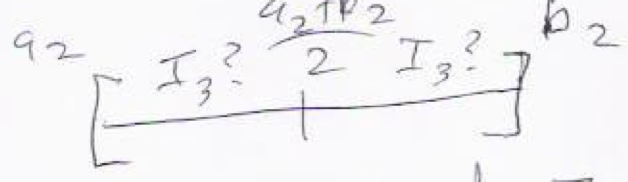
Choose I_1 such that E_1 is infinite. Let $n_1 = \min(E_1)$;
$a_1 = \min(I_1)$ & $b_1 = \max(I_1)$.

$E_1 - \{n_1\} = \{m \in E_1 : m \neq n_1\}$
is infinite, so either ∞-many
$n \in E_1 - \{n_1\}$ are such that $p_n \in [a_1, \frac{a_1+b_1}{2}]$,
or ∞-many $n \in E_1 - \{n_1\}$ are
such that $p_n \in [\frac{a_1+b_1}{2}, b_1]$, or both.

$I_1 = [a_1, b_1]$



Choose $I_2 = [a_1, \frac{a_1+b_1}{2}]$ or
$I_2 = [\frac{a_1+b_1}{2}, b_1]$
such that $E_2 = \{n \in E_1 - \{n_1\} : p_n \in I_2\}$
is infinite. Let $n_2 = \min(E_2)$,
$a_2 = \min(I_2)$, $b_2 = \max(I_2)$.



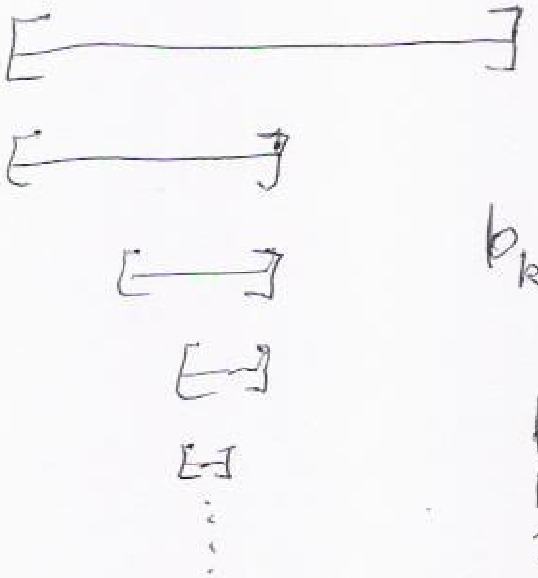
$I_3 = [a_2, \frac{a_2+b_2}{2}]$ or $[\frac{a_2+b_2}{2}, b_2]$
Choose I_3 such that
$E_3 = \{n \in E_2 - \{n_2\} : p_n \in I_3\}$

is infinite. Repeat to define

$I_4, I_5, I_6, \ldots, n_3, n_4, n_5, \ldots$

$E_4, E_5, E_6 \ldots, a_3, a_4, a_5, \ldots,$

$b_3, b_4, b_5, \ldots$ in the same way.



$$b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2}$$

$$\begin{array}{c} b_0 - a_0 = 1 = 2^0 \\ \shortparallel \quad\;\; \shortparallel \\ 1 \quad\;\; 0 \end{array}$$

$\forall n \in E_k \quad p_n \in I_k = [a_k, b_k]$

$$b_1 - a_1 = \frac{1}{2} = 2^{-1}$$

$\forall n \in E_k \quad a_k \le p_n \le b_k$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{1}{4} = 2^{-2}$$

$n_k = \min(E_k)$

$$b_k - a_k = 2^{-k}$$

$$b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2} = 2^{-(k+1)}$$

$n_k \notin E_{k+1}$

$n_k < \min(E_{k+1}) = n_{k+1}$

$\rightarrow n_1 < n_2 < n_3 < n_4 < \cdots$

$$E_{k+1} \subset E_k - \{n_k\} \subset E_k$$

$$n_{k+1} = \min(E_{k+1}) \in E_{k+1} \subset E_k$$

$$n_{k+2} \in E_{k+2} \subset E_{k+1} \subset E_k$$

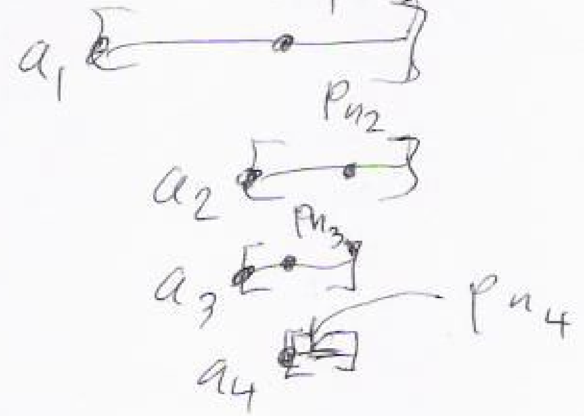
$$\forall \ell \geq k \qquad n_\ell \in E_k, \text{ so } p_{n_\ell} \in I_k$$

$$\forall \ell \geq k \qquad a_k \leq p_{n_\ell} \leq b_k = a_k + 2^{-k}$$

$$\forall x, y \in I_k \quad d(x,y) \leq d(a_k, b_k) = 2^{-k}$$

Let

$$s = \sup\{a_k : k = 1, 2, 3, \ldots\}$$



$$\forall \ell \geq k$$

$$a_k \leq p_{n_\ell} \leq a_k + 2^{-k}$$

$a_1 \le a_2 \le a_3 \le a_4 \le \cdots \le b_4 \le b_3 \le b_2 \le b_1$

Every b_m is an upper bound

of $\{a_k : k = 1, 2, 3, \ldots\}$.

s is the least upper bound,

so $a_1 \le a_2 \le a_3 \le \cdots \le s \le \cdots \le b_3 \le b_2 \le b_1$

$\forall k \ \forall l \ge k \quad p_{n_l}, s \in [a_k, b_k]$

☆ $\forall k \ \forall l \ge k \quad d(p_{n_l}, s) \le d(a_k, b_k) = 2^{-k}$

"$\lim_{k\to\infty} p_{n_k} = s$" means $\forall \varepsilon > 0$

$\exists N_\varepsilon \ \forall l \ge N_\varepsilon \quad d(p_{n_l}, s) < \varepsilon$.

To prove $\lim_{k\to\infty} p_{n_k} = s$, let $\varepsilon > 0$.

Then choose $k > -\log_2(\varepsilon)$

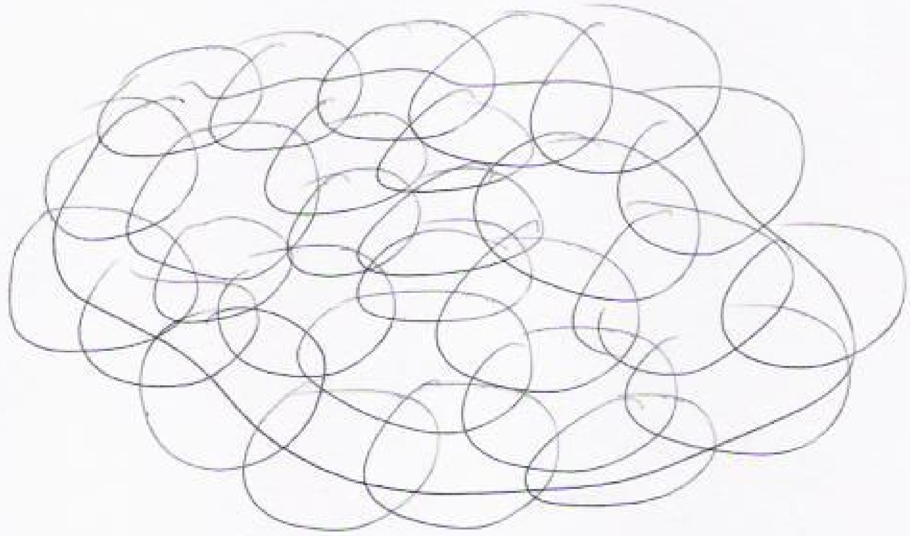
(Solve $2^{-k} < \varepsilon$ for k.)

Then $0 < 2^{-k} < \varepsilon$. Set $N_\varepsilon = k$.

$\forall l \geq N_\varepsilon (=k) \quad d(p_{n_l}, s) \leq 2^{-k} < \varepsilon,$

as desired. $\square$

General argument for compact metric spaces goes through 2.20, which say every infinite set A in a compact metric space has a limit point.



Given a finite open cover $U_1, \dots, U_n$, $U_k \cap A$ is infinite for at least one k.

Then repeat with an open cover

of $\overline{U_k}$, which is compact because it's closed in the compact set X.

The mth open cover should be a finite cover by open balls of radius 2^{-m} (or $1/m$).

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sets $\neq$ sequences

set $\{0, 1, 0, 1, 2, 2, 2, 2\} = \{0, 1, 2\}$

seq. $(0, 1, 0, 1, 2, 2, 2, 2) \neq (0, 1, 2)$

The range of $\cancel{\{a_1, \dots, a_n\}}$

$(a_1, \dots, a_n)$ is $\{a_1, \dots, a_n\}$

"(un)bounded sequence"

= "sequence with (un)bounded range"

"<u>Definition</u> A is B if C."

means $(A \text{ is } B) \Longleftrightarrow C$.

"Theorem/Lemma/Proposition/Fact/Corollary/etc. A is B if C."

means $(A \text{ is } B) \Longleftarrow C$.

"Thm/Lem/Prop/etc

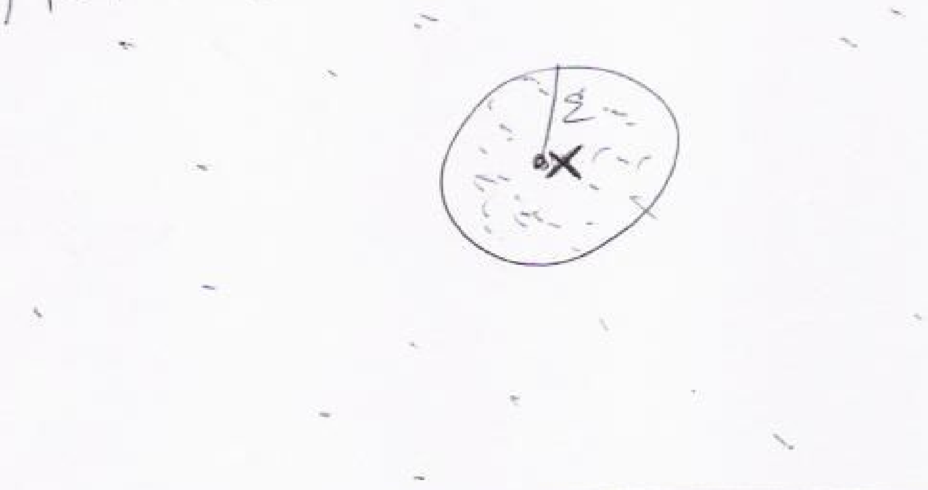
A is B if and only if C."

means $(A \text{ is } B) \Longleftrightarrow C$.

Informal description of $\lim\limits_{n\to\infty} p_n = x$:

For every desired accuracy level, the sequence eventually commits to equalling x ($\pm$ accuracy level).

Picture:



$\forall \varepsilon > 0$

Eventually, it stops leaving the inside of the disc.

Prove that if $\lim\limits_{n\to\infty} a_n = s$ & $\lim\limits_{n\to\infty} b_n = t$ & $\forall n \ b_n \neq 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{s}{t}.$$

Use 3.3(d): $\forall (x_n)_{n=1}^{\infty} \left[\left[\lim\limits_{n\to\infty} x_n = y \ \& \ \forall n \ x_n \neq 0\right] \Rightarrow \lim\limits_{n\to\infty} \frac{1}{x_n} = \frac{1}{y}\right]$

Therefore, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{t}$.

3.3(c): $\forall (x_n)_{n=1}^{\infty} \forall (y_n)_{n=1}^{\infty}$

$[[\lim_{n\to\infty} x_n = z \ \& \ \lim_{n\to\infty} y_n = w]$

$\Rightarrow \lim_{n\to\infty} (x_n y_n) = zw]$

Therefore, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \left(\frac{1}{b_n}\right)$

$= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} \frac{1}{b_n}\right) = s\left(\frac{1}{t}\right) = \frac{s}{t}$. ✓

Prove that if $\forall n \ a_n, b_n, c_n \in \mathbb{R}$

$\& \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = x \in \mathbb{R}$,

$\&$ $a_n \le b_n \le c_n$, then

$\lim_{n\to\infty} b_n = x$ too.

Similarly to proof of 3.3,

given $\varepsilon > 0$, choose M, N

such that $\begin{cases} \forall k \geq M & d(a_n, x) < \varepsilon \\ \forall k \geq N & d(c_n, x) < \varepsilon. \end{cases}$

Let $L = \max(M, N)$.

$\forall k \geq L$, we have $d(b_n, x) < \varepsilon$

because $a_n \leq b_n \leq c_n$ &

Case $x < b_n$:

$x < b_n \leq c_n \Rightarrow d(b_n, x) < d(c_n, x) < \varepsilon$;

Case $b_n \leq x$:

$a_n \leq b_n \leq x \Rightarrow d(b_n, x) \leq d(a_n, x) < \varepsilon$.

$\square$