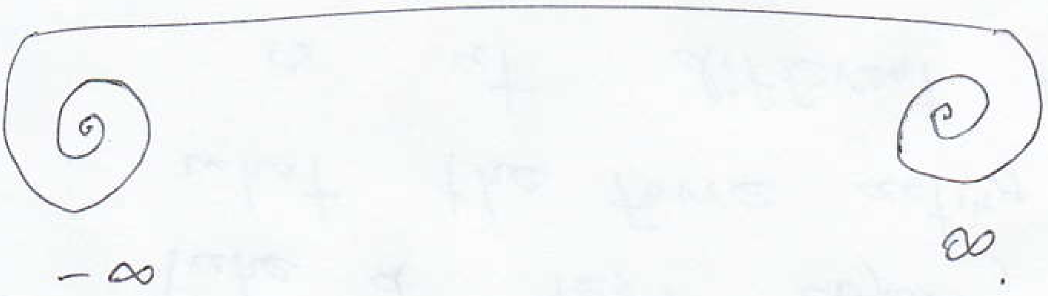
Intuition:

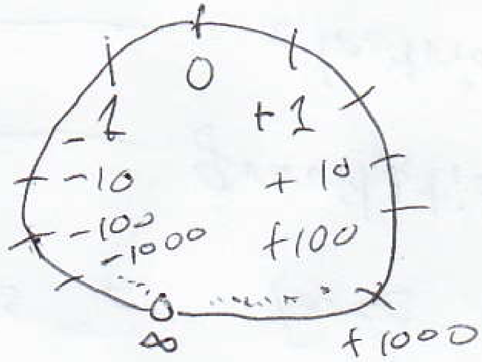
complete = no ~~holes~~ holes, except maybe at ∞

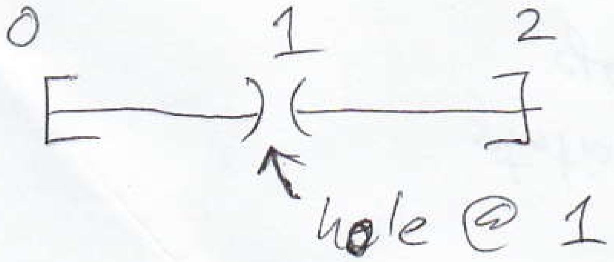
compact = no holes

$\mathbb{R}$ complete but not compact



$\mathbb{R}$





$X = [0,1) \cup (1,2]$ not complete

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ diverges in X.

$\forall E \subset \mathbb{R}^k$

① (E compact $\iff$ E closed & bounded)

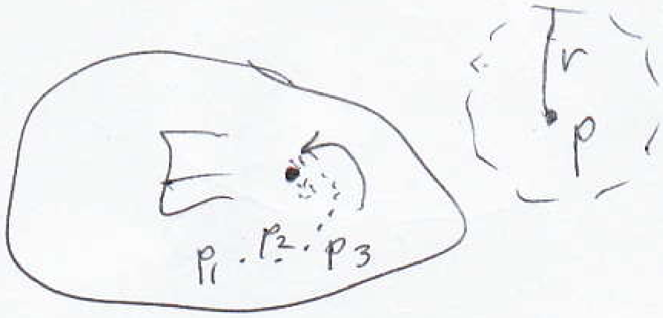
② (E complete $\iff$ E closed)

$\forall (X,d)$ metric space $\quad \forall E \subset X$

E closed $\iff$ $\forall$ convergent $p_1, p_2, p_3, \dots \in E$

$$\lim_{n\to\infty} p_n \in E$$

E closed $\iff$ $\forall p \in X \left[p \notin E \Rightarrow \exists r > 0 \; N_r(p) \cap E = \emptyset \right]$



In $\mathbb{R}^k$, a sequence converges iff it is Cauchy.

(Same if $\mathbb{R}^k$ replaced by any complete metric space.)

Exercise #1-5 of Ch. 3
relevant to Test I topics,
as ~~...~~ are Ch. 1 & 2 exercises.

Let's look at #3 now:

$$s_1 = \sqrt{2} \qquad s_{n+1} = \sqrt{2 + \sqrt{s_n}}$$

$$s_2 = \sqrt{2 + \sqrt{\sqrt{2}}} \qquad s_3 = \sqrt{2 + \sqrt{\sqrt{2 + \sqrt{\sqrt{2}}}}}$$

$$s_4 = \sqrt{2 + \sqrt{\sqrt{2 + \sqrt{\sqrt{2 + \sqrt{\sqrt{2}}}}}}}$$

Prove $s_1, s_2, s_3, \ldots$ converges
and that $\forall n = 1, 2, 3, \ldots \quad s_n < 2$.

Let's prove $\forall n \quad 0 < s_n < 2$
by induction. Case $n=1$: $0 < \sqrt{2} < 2$ ✓.
$(0^2 < 2 < 2^2)$

Assume case $n=k$ & prove case $n=k+1$.

Given: $0 < s_k < 2$. Goal: $0 < \sqrt{2+\sqrt{s_k}} < 2$

$$0 < s_k \Rightarrow 0 < \sqrt{s_k} \Rightarrow 2 < 2 + \sqrt{s_k}$$

$$\Rightarrow \sqrt{2} < \sqrt{2 + \sqrt{s_k}} \; (= s_{k+1}) \Rightarrow 0 < s_{k+1}$$

Note special case: $s_1 = \sqrt{2} < s_2$.

$s_{k+1} = \sqrt{2+\sqrt{s_k}} < 2 \iff 2+\sqrt{s_k} < 4.$

$\iff \sqrt{s_k} < 2 \iff s_k < 4 \Leftarrow s_k < 2$

Done: by induction $\forall n \; 0 < s_n < 2$.

Next step: show $\forall n \; s_n < s_{n+1}$.

For intuition, graph $\sqrt{2+\sqrt{x}}$

We want $\underbrace{x}_{s_n} < \underbrace{\sqrt{2+\sqrt{x}}}_{s_{n+1}}$ when $x = s_n$

Let's graph $y = \sqrt{2+\sqrt{x}}$ & $y = x$.

It appears that near $x = 1.84$

$x = \sqrt{2+\sqrt{x}}$, and $0 < x \underbrace{\lesssim 1.84}_{\text{the cut-off is not exactly } 1.84} \Rightarrow x < \sqrt{2+\sqrt{x}}$.

$\downarrow$

solve: $x^2 = 2 + \sqrt{x}$

$x^2 - 2 = \sqrt{x}$

$(x^2-2)^2 = x$

$x^4 - 4x^2 + 4 = x$

$x^4 - 4x^2 - x + 4 = 0$

$s_{n+1} = \sqrt{2+\sqrt{s_n}}$ & $s_n < 2$

$\Rightarrow s_{n+1} > \sqrt{s_n + \sqrt{s_n}} > \sqrt{s_n}$ (not good enough)

$s_{n+1} > s_n \iff s_{n+1} - s_n > 0$

$s_{n+1} - s_n = \sqrt{2+\sqrt{s_n}} - \sqrt{2+\sqrt{s_{n-1}}}$

$\forall n \geq 2 \quad s_{n+1} > s_n \iff \sqrt{2+\sqrt{s_n}} - \sqrt{2+\sqrt{s_{n-1}}} > 0$

Simpler: $s_{n+1} > s_n \iff \sqrt{2+\sqrt{s_n}} > \sqrt{2+\sqrt{s_{n-1}}}$

Proof that $\forall n \; s_{n+1} > s_n$:

Case $n=1$: $s_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = s_1$ ✓

Assume case $n=k$: $s_{k+1} > s_k$.

Then $\sqrt{s_{k+1}} > \sqrt{s_k}$.

Then $2 + \sqrt{s_{k+1}} > 2 + \sqrt{s_k}$.

Finally, $\sqrt{2+\sqrt{s_{k+1}}} > \sqrt{2+\sqrt{s_k}}$

So, $s_{k+1+1} > s_{k+1}$,

that is case $n=k+1$ follows.

So, $(s_n)_{n=1}^{\infty}$ is bounded & monotonic; and therefore convergent in $\mathbb{R}$. (3.14)

$$\sqrt{2} < \sqrt{2+\sqrt{\sqrt{2}}} < \sqrt{2+\sqrt{\sqrt{2+\sqrt{\sqrt{2}}}}} < \cdots < 2$$

More advanced: prove that $(s_n)_{n=1}^{\infty}$ converges to the unique solution to $\sqrt{2+\sqrt{x}} = x$.