1. Fill in the blanks with yes/no

$X \subset \mathbb{R}$	open	closed	compact	connected	perfect	bounded	countable
$[2,3)$							
$\{1,2,3\}$							
$(-\infty,0]$							
$\mathbb{R}$							
$\mathbb{Q}$							
$(-1,0)\cup(0,1)$							
$(-1,1)$							
$\{1,4,9,16,\dots\}$							
$[5,7]$							
$[1,2]\cup[3,4]$							
$(0,\infty)$							
$\{1\}\cup[2,3]$							

2. Prove that every connected subset of $\mathbb{R}$ is convex.

3. If $(a+bi) \boxtimes (c+di) = ac + (ad+bc)i$ for all $a, b, c, d \in \mathbb{R}$, is $(\mathbb{C}, +, \boxtimes)$ a field? Justify your answer.

4. Prove that if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{R}$, then $((x_n, y_n, x_n + y_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^3$.

5. Give an example of a countable dense subset of $\mathbb{R}^5$.

Remarks: ① the actual test need not match the practice test's choice of topics. ② On the actual test, you are permitted notes in your own handwriting (which you must turn in with your test), but it is closed-book.
③ The questions here are a rough estimate of the difficulty level of the actual test.

open closed perfect dense bounded

ctbl/unctbl/finite | cpct | lim pts

fields | metric space | connected | int pts

order | convex | iso pts

$\mathbb{C}$, $\mathbb{R}^k$

sup

$\cap, \cup, {}^c$

convergent, Cauchy, monotonic, bounded

subseq

#1 $\mathbb{Q}$ not open.

If $a, b \in \mathbb{Q}$ & $a < b$, then

$$a < \frac{a + \sqrt{2}\, b}{1+\sqrt{2}} < b \quad \& \quad \frac{a+\sqrt{2}\, b}{1+\sqrt{2}} \notin \mathbb{Q}$$

$$\frac{y}{x} = \sqrt{2} \notin \mathbb{Q}$$

$$c \in \mathbb{Q} \Rightarrow \frac{y}{x} = \frac{b-c}{c-a} \in \mathbb{Q}$$

$$\Rightarrow c \notin \mathbb{Q}$$

$\therefore (a,b) \not\subset \mathbb{Q}$

$(0,0),\ (0,1),\ (1,0),\ (1,1),$
$(0,2),\ (1,2),\ (2,0),\ (2,1),\ (2,2),$
$(0,3), \ldots$ lists $\{0,1,2,\ldots\}^2$

$\frac{0}{1},\ \frac{1}{1},\ -\frac{1}{1},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{2}{1},\ -\frac{2}{1},$
$\frac{1}{3},\ -\frac{1}{3}, \ldots$ lists $\mathbb{Q}$

E ~~is~~ disconnected iff $\exists A, B \subset E$

$A, B \neq \emptyset$ & $A \cap \bar{B} = \emptyset = \bar{A} \cap B$

& $E = A \cup B$

$A = [0;1) \quad B = (1,2]$

$E = A \cup B = [0,1) \cup (1,2]$

is disconnected

$\bar{A} = [0,1] \quad \bar{B} = [1,2]$

If $E \subset \mathbb{R}$, then E disconn

iff $\exists a, b \in E \quad \exists c \notin E$

$a < c < b$. If this

happens, then $A = E \cap (-\infty, c)$

$B = E \cap (c, \infty)$

witness E disconn.

~~E convex $\subset \mathbb{R} \Rightarrow E$ connected $\subset \mathbb{R}$~~

~~Same as:~~

#2

E connected $\subset \mathbb{R} \Rightarrow E$ convex $\subset \mathbb{R}$

Same as:

[$E \subset \mathbb{R}$ not convex $\Rightarrow E$ not connected]

Proof: If E not convex, then $\exists\, a, b \in E$ $\exists\, c \in E^c$ $a < c < b$.

Let ~~$A = (-\infty, c)$ & $B = (c, \infty)$.~~

~~$\bar{A} = (-\infty, c] \cap E$ & $\bar{B} = [c, \infty)$~~

~~$a \in A$~~

$A = (-\infty, c) \cap E \ni a$

$B = (c, \infty) \cap E \ni b$

$\bar{A} \subset (-\infty, c]$ $\bar{B} \subset [c, \infty)$

So $\bar{A} \cap B = \bar{B} \cap A = \emptyset$

& $A \cup B = E$ & $A, B \neq \emptyset$. ✓

#3.

$(a+bi) + (c+di) = (a+c) + (b+d)i$

$(a+bi) \times (c+di) = (ac-bd) + (ad+bc)i$

$(a+bi) \boxtimes (c+di) = ac \quad + (ad+bc)i$

$(\mathbb{C}, +, \times)$ is a field

$(\mathbb{C}, +, \boxtimes)$ a field? No: $i^2 = 0$:

Since $+$ is unchanged, 0 is the same as usual, that is,

$\forall z \in \mathbb{C} \quad 0+z = z+0 = z$ &

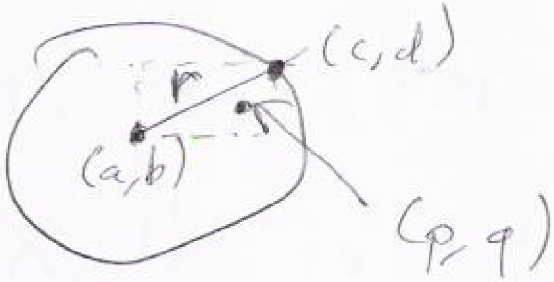
$\forall w \neq 0 \ \forall z \in \mathbb{C} \quad z+w \neq z \neq w+z$.

In particular, $i \neq 0$, so i^{-1} exists, if $(\mathbb{C}, +, \boxtimes)$ is a field.

i^{-1} exists $\Longrightarrow i^{-1} i^2 = i^{-1} i\, i = 1 i = i$

But $i^{-1} i^2 = i^{-1} 0 = 0 \neq i$, so

$(\mathbb{C}, +, \boxtimes)$ is not a field.



$a < c \Rightarrow \exists_{p} \in \mathbb{Q} \cap (a,c)$

$b < d \Rightarrow \exists_{q} \in \mathbb{Q} \cap (b,d)$

$\mathbb{Q}^2$ is dense in $\mathbb{R}^2$,

i.e., $\forall r > 0 \ \forall (s,t) \in \mathbb{R}^2$

$\exists (p,q) \in \mathbb{Q}^2 \ d((p,q),(s,t)) < r$

#5

Similarly, $\mathbb{Q}^5$ is dense in $\mathbb{R}^5$

$\mathbb{Q} = \{q_1, q_2, q_3, q_4, \ldots\}$

$\mathbb{Q}^5 = \{(q_1, q_1, q_1, q_1, q_1),$

$(q_2, q_1, q_1, q_1, q_1),$

$(q_1, q_2, q_1, q_1, q_1), \ldots$

$p_1, p_2, p_3, p_4, \ldots$ in a metric space

(X, d) is Cauchy iff

$\forall \varepsilon > 0$ $(p_n)_{n=1}^{\infty}$ eventually commits

to varying less than ε, that is,

$\exists N_\varepsilon \ \forall m, n \geq N_\varepsilon \ d(p_m, p_n) < \varepsilon$.

~~#6~~ #4

Given: $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ Cauchy

in $\mathbb{R}$ where $d(x, y) = |x - y|$.

Prove $((x_n, y_n, x_n + y_n))_{n=1}^{\infty}$

Cauchy in $\mathbb{R}^3$ where

$d((a_1, a_2, a_3), (b_1, b_2, b_3))$

$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$

Let $\varepsilon > 0$. So, $\exists L, M$ such

that $\forall m, n \geq L$ $|x_m - x_n| < \varepsilon/100$

& $\forall m, n \geq M$ $|y_m - y_n| < \varepsilon/100$

Let $N = \max(L, M)$.

$$\forall\, m,n \geq N \quad |x_m - x_n|,\ |y_m - y_n| < \frac{\varepsilon}{100}$$

$$z_n = (x_n, y_n, x_n + y_n)$$

$$d(z_m, z_n) = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2 + (x_m + y_m - x_n - y_n)^2}$$

$$< \sqrt{(\varepsilon/100)^2 + (\varepsilon/100)^2 + (2\varepsilon/100)^2}$$

$$= \sqrt{(1 + 1 + 2^2)(\varepsilon/100)^2}$$

$$= \sqrt{6}\,\varepsilon/100 < \varepsilon \checkmark$$