More review exercises

domain = codomain = $\mathbb{R}$ in all examples below

I) True or False?

1) Continuous images of compact sets are compact, always. True 4.14

2) Continuous images of compact sets are closed, always. True : compact $\Downarrow$ closed

3) Continuous images of closed sets are closed, always. False: arctan(x)

4) Continuous preimages of closed sets are closed, always True

5) Continuous preimages of open sets are open, always True

6) Continuous images of open sets are open, always. False $(-\infty,\infty) \xrightarrow{\text{squaring}} [0,\infty)$ $x \longmapsto x^2$
sin(x)

7) Continuous preimages of compact sets are compact, always. False $\frac{x}{1+x^2}$ or arctan(x)
sin(x)

8) Continuous images of connected sets are connected, always True 4.22

9) Continuous preimages of connected sets are connected, always.

False sin(x)

II) For each "False" answer, give a counterexample.

III) Which of these series converge?

3.28 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

3.42 or 3.44 $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n\sqrt{n}}$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}}$

3.26 $\displaystyle\sum_{n=0}^{\infty} 2^{-n}$ $\quad \displaystyle\sum_{n=0}^{\infty} n!\,2^{-n}$ (3.23) $\quad \displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ (3.40 b)

IV Which of these functions are uniformly continuous on the domain $(0,\infty)$?

$x \mapsto x^2$ $\quad x \mapsto x$ $\quad x \mapsto \sqrt{x}$

$x \mapsto e^x$ $\quad x \mapsto \frac{1}{x^2+1}$ $\quad x \mapsto \frac{1}{x}$

$x \mapsto e^{-x}$ $\quad x \mapsto \sin\left(\frac{1}{x}\right)$ $\quad x \mapsto 5$

$(e^{-x})' = -e^{-x}$ bounded on $(0,\infty)$

$(\sqrt{x})'$ is bounded on $(1,\infty)$

Throw away compact $[0,1]$ on which $\sqrt{\ }$ is unif cts

Solution to **V**. Let $b_k = \sup\{a_n : n \geq k\}$.

Then $b_1 \geq b_2 \geq b_3 \geq \cdots$ & $b_k \to b$

where $b = \limsup_{n\to\infty} a_n$.

Case $-\infty < b < \infty$: Set $n_1 = 1$.

Given n_l for all $l < k$,

choose n_k as follows: Since $b_1, b_2, \ldots \to b$, $\exists N\ \forall m \geq N\ |b_m - b| < \frac{1}{k}$

Let $m = \max(N, n_{k-1} + 1) > n_{k-1}$

Then $b \leq b_m < b + \frac{1}{k}$. Since b_m is the least upper bound of $\{a_h : h \geq m\}$, $b_m - \frac{1}{k}$ is not a upper bound of $\{a_h : h \geq m\}$ so choose $n_k \geq m$ such that $b_m - \frac{1}{k} < a_{n_k}$

Then $b - \frac{1}{k} \leq b_m - \frac{1}{k} < a_{n_k} \leq b_m < b + \frac{1}{k}$,

so, as $k \to \infty$, $a_{n_k} \to b$. (Note: $n_k \geq m > n_{k-1}$

Case $b = -\infty$: Given $n_1, \dots, n_{k-1}$, choose n_k as follows: Since $b_1, b_2, b_3, \dots \to -\infty$, $\exists N\ \forall m \geq N$ $b_m \leq -k$. Let $m = \max(N, n_{k-1} + 1) > n_{k-1}$. Since b_m is an upper bound of $\{a_h : h \geq m\}$, let $n_k = m > n_{k-1}$, so that $a_{n_k} \leq -k$.

Hence, as $k \to \infty$, $a_{n_k} \to -\infty$.

Case $b = \infty$: Given $n_1, \dots, n_{k-1}$, choose n_k as follows: Since $b_1 = b_2 = b_3 = \dots = \infty$ (because $b_1 \geq b_2 \geq \dots \geq b$), ∞ is the least upper bound of $\{a_h : h \geq n_{k-1}\}$, so k is not an upper bound of $\{a_h : h \geq n_{k-1}\}$, so choose $n_k > n_{k-1}$ such that $a_{n_k} > k$. Hence, $a_{n_k} \to \infty$ as $k \to \infty$. $\square$

Solution VI:

If $a_n \to b$ in a metric space X, then $\forall \varepsilon > 0 \ \exists N_\varepsilon \ \forall m \geq N_\varepsilon \ d(a_m, b) < \varepsilon$

This implies that $\forall \varepsilon > 0$ $\forall m, n \geq N_{\varepsilon/2} \ d(a_m, a_n) \leq d(a_m, b) + d(b, a_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $(N_{\varepsilon/2} : \varepsilon > 0)$ witnesses that $(a_n)_{n=1}^{\infty}$ is Cauchy.

Solution VII There must exist $p < q < r$ in $[a, b]$ such that $f(p) < f(q) > f(r)$ or $f(p) > f(q) < f(r)$. ~~By symmetry~~ The second case follows from the first case applied to $-f$, so assume the first case holds. ~~Let $y = \max(f(p), f(r))$.~~ ~~By the Intermediate Value Theorem, $f(s) = y$ for some s in $[p, q]$ & $f(t) = y$ for some t in $[q, r]$.~~ By ~~Rolle's Theorem~~ the Extreme Value Theorem, $f(u) = \max\{f(x) : p \leq x \leq r\}$ for some u in $[p, r]$, and actually $p < u < r$ because $f(u) \geq f(q) > f(p), f(r)$. Letting $\delta = \min(u-p, r-u)$, we have $[d(u,x) < \delta \Rightarrow f(x) \leq f(u)]$ for all $x \in \mathbb{R}$, so f has a local max. at u, so $f'(u) = 0$ by 5.8. $\square$