($\star$) $E \subset X$ is compact iff

$\forall p_1, p_2, p_3, \ldots \in E \quad \exists q \in E$

$\exists n_1 < n_2 < n_3 < \cdots \quad \lim_{k \to \infty} p_{n_k} = q.$

(In other words, E is compact iff every sequence in E has a subsequence that converges in E.)

Alternative proof that if $f: X \to Y$ cts & $E \subset X$ compact, then $f(E)$ is compact:

Suppose $p_1, p_2, p_3, \ldots \in f(E)$.

For each n, choose $x_n \in E$ such that $f(x_n) = p_n$. By compactness of E & ($\star$), $\exists n_1 < n_2 < n_3 < \cdots \quad \exists q \in E$

$\lim_{k \to \infty} x_{n_k} = q$. By continuity of f, $\lim_{k \to \infty} p_{n_k} = f(q)$

and $f(q) \in f(E)$ because $q \in E$.

Therefore, by $(\star)$, $f(E)$ is compact. $\square$

$(\star\star)$ $E \subset X$ is connected iff for every nonempty partition of E into two nonempty sets A, B, there exist $p_1, p_3, p_5, \ldots \in A$, $p_2, p_4, p_6, \ldots \in B$, and $q \in E$ such that $\lim_{n \to \infty} p_n = q$.

Alternative proof that if $f: X \to Y$ cts. & $E \subset X$ is connected, then $f(E)$ is connected:

Suppose A, B are nonempty & disjoint, and $f(E) = A \cup B$. Then let

$$\begin{cases} C = E \cap f^{-1}A = \{p \in E : f(p) \in A\} \\ D = E \cap f^{-1}B = \{p \in E : f(p) \in B\}. \end{cases}$$

This makes C & D nonempty and disjoint, and $E = C \cup D$. By $(\star\star)$,

$\exists q \in E \;\; \exists p_1, p_3, p_5, \ldots \in C$
$\exists p_2, p_4, p_6, \ldots \in D \;\; \lim_{n \to \infty} p_n = q.$

By continuity of f, $\lim_{n \to \infty} f(p_n) = f(q)$.

Moreover, $f(q) \in f(E)$, and, by def. of C & D, $f(p_1), f(p_3), f(p_5), \ldots \in A$, and $f(p_2), f(p_4), f(p_6), \ldots \in B$. By (**), $f(E)$ is connected. $\square$

If you like sequences a lot, you might like these characterizations:

- $E \subset X$ is open iff $\forall p_1, p_2, p_3, \ldots \in X$ [if $\lim_{n \to \infty} p_n$ exists and is in E, then $\exists N \; \forall n \geq N \; p_n \in E$].
- $E \subset X$ is closed iff $\forall p_1, p_2, p_3, \ldots \in E$ [if $\lim_{n \to \infty} p_n$ exists, then it is in E].