MATH 4335 Homework 23

Given $f\colon [a,b] \to \mathbb{R}$, define the total variation of f, which is also denoted by $\int_a^b |df|$, as the supremum of all sums of the form

$|f(p_1)-f(p_0)|+|f(p_2)-f(p_1)|+|f(p_3)-f(p_2)|+\cdots+|f(p_n)-f(p_{n-1})|$

for which $(p_0, p_1, p_2, \ldots, p_n)$ is a partition of $[a,b]$.

Suppose that $M \in [0,\infty)$ and, for all $x \in [a,b]$, $f'(x)$ exists and satisfies $|f'(x)| \le M$. Prove that then

$$\int_a^b \sqrt{df^2 + dx^2} \le \int_a^b |df| + b - a \le M(b-a) + b - a.$$

Hint:

$$\sqrt{\alpha^2+\beta^2} \le \sqrt{\alpha^2 + 2|\alpha||\beta| + \beta^2} = \sqrt{(|\alpha|+|\beta|)^2}.$$

Warning: Here we are not assuming that f' is continuous.