As a special case of linearity of infinite series, if $\sum_{m=0}^{\infty} a_m$ and $\sum_{m=0}^{\infty} b_m$ are convergent, then $\sum_{m=0}^{\infty}(a_m + b_m)$ converges to $(\sum_{m=0}^{\infty} a_m) + (\sum_{m=0}^{\infty} b_m)$. Using induction, it is easy to prove that, for any integer $k \geq 0$, if $\sum_{m=0}^{\infty} a_{m,n}$ converges for each integer n from 0 to k, then $\sum_{m=0}^{\infty}\left(\sum_{n=0}^{k} a_{m,n}\right)$ converges to $\sum_{n=0}^{k}\left(\sum_{m=0}^{\infty} a_{m,n}\right)$. The following theorem is a generalization to $k = \infty$.

Lemma. *If $\sum a_n$ is absolutely convergent, then $|\sum a_n| \leq \sum |a_n|$.*

Proof. First, $\sum a_n$ converges since it is absolutely convergent. Second, by the triangle inequality, every partial sum $\sum_{n \leq N} a_n$ satisfies

$$\left|\sum_{n \leq N} a_n\right| \leq \sum_{n \leq N} |a_n| \leq \sum |a_n|.$$

Hence, by the Limit Location Theorem, $|\sum a_n| \leq \sum |a_n|$. □

Theorem. *Suppose that $x_{m,n} \in \mathbb{R}$ for all $m, n \geq 0$, that $\sum_{m=0}^{\infty} |x_{m,n}|$ converges for all $n \geq 0$, and that $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} |x_{m,n}|\right)$ converges. Then $\sum_{n=0}^{\infty} |x_{m,n}|$ converges for all $m \geq 0$, all the series in the equation*

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} x_{m,n}\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} x_{m,n}\right)$$

converge, and that equation is true.

Proof. We first show, for each m, that $\sum_{n=0}^{\infty} |x_{m,n}|$ converges. Since each term $|x_{m,n}|$ is nonnegative, it is enough to show that the partial sums $\sum_{n=0}^{N} |x_{m,n}|$ has an upper bound. The sum $\sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty} |x_{i,n}|\right)$, which is finite by hypothesis, is such a bound:

$$\sum_{n=0}^{N} |x_{m,n}| \leq \sum_{n=0}^{N}\left(\sum_{i=0}^{\infty} |x_{i,n}|\right) \leq \sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty} |x_{i,n}|\right).$$

Since each $\sum_{n=0}^{\infty} x_{m,n}$ is, therefore, absolutely convergent, it is convergent. Likewise, since each $\sum_{m=0}^{\infty} x_{m,n}$ is, by hypothesis, absolutely convergent, it is convergent. Let $a_m = \sum_{n=0}^{\infty} x_{m,n}$ for each m and let $b_n = \sum_{m=0}^{\infty} x_{m,n}$ and $c_n = \sum_{m=0}^{\infty} |x_{m,n}|$ for each n. By the lemma, $|b_n| \leq c_n$ for each n. By hypothesis, $C = \sum_{n=0}^{\infty} c_n$ is convergent; hence, every partial sum $\sum_{n=0}^{N} |b_n|$ is bounded above by C; hence, $\sum_{n=0}^{\infty} b_n$ is absolutely convergent and, therefore, convergent. Let $B = \sum_{n=0}^{\infty} b_n$.

It remains to show that $\sum_{m=0}^{\infty} a_m$ converges to B. Given $\varepsilon > 0$, we will find K such that $\sum_{m=0}^{H} a_m \underset{\varepsilon}{\approx} B$ for all $H \geq K$. It is sufficient to find K large enough that $\sum_{m=0}^{K} \sum_{n=0}^{K} |x_{m,n}| \underset{\varepsilon}{\approx} C$ because, informally speaking, if the magnitudes of every term $x_{m,n}$ outside the upper left $K \times K$ square of the infinite matrix $\{x_{m,n}\}$ add up to less than ε, then the difference between a sum of the top K

or more rows and a sum of the left K or more columns should be smaller than ε.

Towards a formal version of the above idea, for each $H \geq 0$, let $A_H = \sum_{m=0}^{H} a_m$, $B_H = \sum_{n=0}^{H} b_n$, $C_H = \sum_{n=0}^{H} c_n$, $D_H = \sum_{n=0}^{H} \left(\sum_{m=0}^{H} x_{m,n} \right)$, and $E_H = \sum_{n=0}^{H} \left(\sum_{m=0}^{H} |x_{m,n}| \right)$. Observe that, since each $|x_{m,n}|$ is nonnegative, we have $E_0 \leq E_1 \leq E_2 \leq E_3 \leq \cdots$. Since $C_N \to C$, we may choose N large enough that $C_N \geq C - \varepsilon/2$. For each n from 0 to N, since $c_n = \sum_{m=0}^{\infty} |x_{m,n}|$, we may choose k_n large enough that $\sum_{m=0}^{k_n} |x_{m,n}| \geq c_n - \varepsilon/(2^{n+2})$. Let $K = \max\{N, k_0, k_1, k_2, \ldots, k_N\}$. Then, since $\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N} < \frac{1}{2}$,

$$E_K \geq \sum_{n=0}^{N} \left(\sum_{m=0}^{k_n} |x_{m,n}| \right) \geq \sum_{n=0}^{N} \left(c_n - \frac{\varepsilon}{2^{n+2}} \right) > C_N - \frac{\varepsilon}{2} \geq C - \varepsilon.$$

Now, suppose $H \geq K$. We then have $E_H \geq E_K > C - \varepsilon$. Moreover,

$$\begin{aligned}
|B - A_H| &= \left| \left(B_H + \sum_{n=H+1}^{\infty} b_n \right) - \left(D_H + \sum_{m=0}^{H} \left(\sum_{n=H+1}^{\infty} x_{m,n} \right) \right) \right| \\
&= \left| (B_H - D_H) + \left(\sum_{n=H+1}^{\infty} b_n \right) - \sum_{n=H+1}^{\infty} \left(\sum_{m=0}^{H} x_{m,n} \right) \right| \\
&= \left| \sum_{n=0}^{H} \left(\sum_{m=H+1}^{\infty} x_{m,n} \right) + \sum_{n=H+1}^{\infty} \left(b_n - \sum_{m=0}^{H} x_{m,n} \right) \right| \\
&= \left| \sum_{n=0}^{H} \left(\sum_{m=H+1}^{\infty} x_{m,n} \right) + \sum_{n=H+1}^{\infty} \left(\sum_{m=H+1}^{\infty} x_{m,n} \right) \right| \\
&\leq \left| \sum_{n=0}^{H} \left(\sum_{m=H+1}^{\infty} x_{m,n} \right) \right| + \left| \sum_{n=H+1}^{\infty} \left(\sum_{m=H+1}^{\infty} x_{m,n} \right) \right| \\
&\leq \sum_{n=0}^{H} \left| \sum_{m=H+1}^{\infty} x_{m,n} \right| + \sum_{n=H+1}^{\infty} \left| \sum_{m=H+1}^{\infty} x_{m,n} \right| \\
&\leq \sum_{n=0}^{H} \left(\sum_{m=H+1}^{\infty} |x_{m,n}| \right) + \sum_{n=H+1}^{\infty} \left(\sum_{m=H+1}^{\infty} |x_{m,n}| \right) \\
&\leq \sum_{n=0}^{H} \left(\sum_{m=H+1}^{\infty} |x_{m,n}| \right) + \sum_{n=H+1}^{\infty} c_n \\
&= (C_H - E_H) + (C - C_H) \\
&= C - E_H \\
&< \varepsilon.
\end{aligned}$$

□