$$\sum_{1}^{n} \frac{1}{n^2+k} \longrightarrow 0$$

Proof. $\displaystyle\sum_{1}^{n} \frac{1}{n^2+k} = \underbrace{\frac{1}{n^2+1} + \cdots \frac{1}{n^2+n}}_{n \text{ terms}}$

$$< \underbrace{\frac{1}{n^2+0} + \cdots + \frac{1}{n^2+0}}_{n} = \frac{n}{n^2}$$

$= \dfrac{1}{n}$. Therefore, given $\varepsilon > 0$,

$$0 < \sum_{1}^{n} \frac{1}{n^2+k} < \frac{1}{n} \leq \varepsilon \text{ for all } n \geq \frac{1}{\varepsilon}.$$

Thus, $\displaystyle\sum_{1}^{n} \frac{1}{n^2+k} \underset{\varepsilon}{\approx} 0$ for all $n \geq 1/\varepsilon$. $\square$

Given $p \in \mathbb{N}$ and $1 < a \in \mathbb{R}$,

$a^n / n^p \to \infty$.

Proof. Let $x = a - 1$. Then $x > 0$.

For all $n \geq p+1$,

$$a^n = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\geq \binom{n}{p+1} x^{p+1}.$$

For all $n \geq 2p$, we have $-\frac{n}{2} \leq -p$

implying ~~...~~

$$\binom{n}{p+1} = \frac{n(n-1)(n-2)\cdots\cdots(n-p)}{1 \cdot 2 \cdot 3 \cdots\cdots (p+1)}$$

$$\geq \frac{n(n-n/2)(n-n/2)\cdots\cdots(n-n/2)}{1 \cdot 2 \cdot 3 \cdots\cdots (p+1)}$$

$$= \frac{n\,(n/2)^p}{(p+1)!} = \frac{n^{p+1}}{2^p\,(p+1)!}.$$

Note: $p \in \mathbb{N} \Rightarrow 1 \leq p \Rightarrow p+1 \leq p+p = 2p$

Hence, $n \geq 2p \Rightarrow n \geq 2p, p+1$

$\Rightarrow a^n \geq n^{p+1} 2^{-p} (p+1)!^{-1}$

$\Rightarrow a^n n^{-p} \geq n \cdot 2^{-p} (p+1)!^{-1}$.

Hence, given $M \in \mathbb{R}$,

$a^n / n^p \geq n / (2^p (p+1)!) \geq M$

for all $n \geq \max\{2p, M \cdot 2^p \cdot (p+1)!\}$. $\square$