5th Homework
Due Day 20 (Oct. 27)

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Definition

1. Given a sequence $\{a_n\}$ and a real U, we say that U is an *eventual upper bound* of $\{a_n\}$ if $a_n \leq U$ for $n \gg 1$.
2. Given a sequence $\{a_n\}$ and a real U, we say U is an *upper limit* of $\{a_n\}$ if, for every $\varepsilon > 0$, $U + \varepsilon$ is an eventual upper bound of $\{a_n\}$ and $U - \varepsilon$ is not an eventual upper bound of $\{a_n\}$.

Use the above definition of upper limit for exercises 1 and 2.

Exercises

1. (a) Prove that if A and B are upper limits of $\{a_n\}$, then $A = B$.
   (b) Prove that if $a_n \to L$, then L is the upper limit of $\{a_n\}$.
   (c) Prove that if $\{a_n\}$ is bounded, then it has an upper limit. (Hint: the upper limit is the greatest lower bound of a certain set...)
2. (a) Prove that if U is a positive eventual upper bound of $\{\sqrt[n]{|a_n|}\}$ and $|x| < 1/U$, then $\sum |a_n x^n|$ converges.
   (b) Prove if if $U > 0$, U is not an eventual upper bound of $\{\sqrt[n]{|a_n|}\}$, and $|x| > 1/U$, then $a_n x^n \not\to 0$.
   (c) Prove that if U is a positive upper limit of $\{\sqrt[n]{|a_n|}\}$, then $\sum a_n x^n$ has radius of convergence $1/U$.
3. (a) Prove by induction that if $k \in \mathbb{N}$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absolutely, then $\sum_{n=0}^{\infty} b_n^{(k)} x^n$ converges to $g(x)^k$ and converges absolutely where $b_n^{(m)}$ is recurvisely defined for all $m, n \in \mathbb{N}$ by $b_0^{(0)} = 1$, $b_n^{(0)} = 0$ if $n > 0$, and
   $$b_n^{(m+1)} = b_0^{(m)} b_n + b_1^{(m)} b_{n-1} + b_2^{(m)} b_{n-2} + \cdots + b_{n-2}^{(m)} b_2 + b_{n-1}^{(m)} b_1 + b_n^{(m)} b_0.$$
   (b) Prove that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has infinite radius of convergence, and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ has positive radius of convergence R, then, for each $n \in \mathbb{N}$, the series $\sum_{m=0}^{\infty} a_m b_n^{(m)}$ coverges absolutely and, for all $x \in (-R, R)$, $\sum_{n=0}^{\infty} c_n x^n$ converges to $f(g(x))$ and converges absolutely, where $c_n = \sum_{m=0}^{\infty} a_m b_n^{(m)}$.