MATH 4335 TEST 2

Name: ______________________________

Date: Oct. 29, 2015.

1. Prove that if a sequence $a_1, a_2, a_3, \ldots$ satisfies $|a_n - a_{n+1}| \leq n^{-3}$ for all n, then $\{a_n\}$ converges.

2.

(a) Explain what is wrong with the "proof" below.
(b) Give a counterexample to the claim.

Claim. *If* $\sum_{i=0}^{\infty} a_i$ *converges and* $b_0, b_1, b_2, \ldots$ *is a subsequence of* $\{a_i\}$*, then* $\sum_{i=0}^{\infty} b_i$ *converges.*

Proof. Given $\varepsilon > 0$, it is enough to show that $b_i + b_{i+1} + b_{i+2} + \cdots + b_j < \varepsilon$ for $j \geq i \gg 1$. By definition of subsequence, we have $b_i = a_{k_i}$ for all i, for some $k_0 < k_1 < k_2 < k_3 < \cdots$. Since $\sum a_i$ converges, there is M such that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n < \varepsilon$ for all $n \geq m \geq M$. Choose N such that $k_N \geq M$. Then $j \geq i \geq N$ implies $k_j \geq k_i \geq M$, which implies

$$b_i + b_{i+1} + b_{i+2} + \cdots + b_j \leq a_{k_i} + a_{k_i+1} + a_{k_i+2} + \cdots + a_{k_j} < \varepsilon. \quad \square$$

3.

(a) Give an example of a sequence with exactly one cluster point.
(b) Give an example of a sequence with exactly three cluster points.
(c) Give an example of a sequence with infinitely many cluster points.
(d) Give an example of a sequence with no cluster points.