4.4 #7: $\sqrt{x}$ is unif. cts. on $[0,\infty)$.

<u>Direct ε-δ proof.</u> Let $\varepsilon > 0$.

If $x, y \geq 0$ & $x \leq \delta$ & $|x-y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \sqrt{2\delta}$ because $x, y \in [0, \delta + \delta)$.

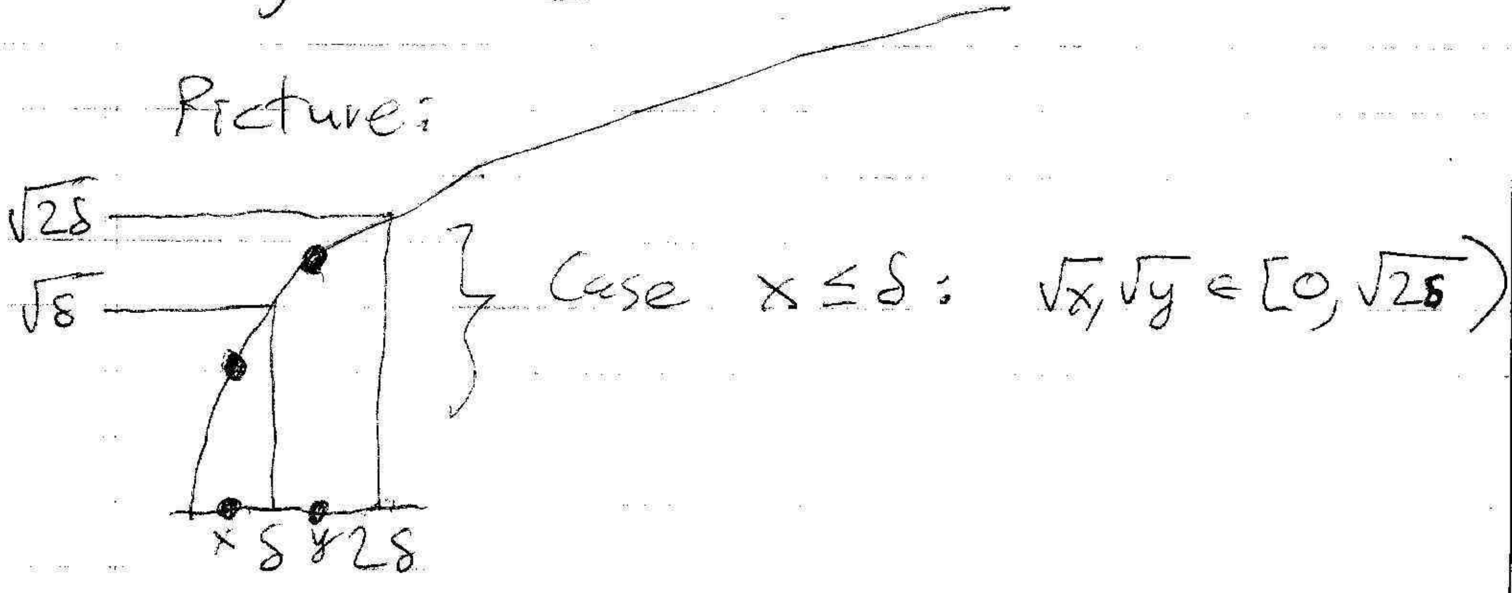
If $x, y \geq 0$ & $x > \delta$ & $|x-y| < \delta$, then $|\sqrt{x} - \sqrt{y}| = \dfrac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \dfrac{|x-y|}{\sqrt{\delta}+0} < \dfrac{\delta}{\sqrt{\delta}}$.

Therefore, it's enough to find $\delta > 0$ such that $\sqrt{2\delta}, \dfrac{\delta}{\sqrt{\delta}} \leq \varepsilon$.

$\delta > 0 \Rightarrow \dfrac{\delta}{\sqrt{\delta}} = \sqrt{\delta} < \sqrt{2\delta}$.

So, $\delta = \frac{1}{2}\varepsilon^2$ works. ■

Picture:



§4.4 #7 Claim. $\sqrt{x}$ is unif. cts. on $[0, \infty)$.

Sequential Proof. $\sqrt{x}$ is cts. on $[0,\infty)$, and, hence, cts. on $[0,1]$. Since $[0,1]$ is compact, $\sqrt{x}$ is unif. cts. on $[0,1]$. Now suppose $x_n, y_n \geq 0$ & $(x_n - y_n) \to 0$. It's enough to prove $(\sqrt{x_n} - \sqrt{y_n}) \to 0$. Suppose $(\sqrt{x_n} - \sqrt{y_n}) \not\to 0$. Then there ∃ a subsequence pair (x_{n_k}) of (x_n) & (y_{n_k}) of (y_n) and $\varepsilon > 0$ such that $|\sqrt{x_{n_k}} - \sqrt{y_{n_k}}| \geq \varepsilon$ for all k. Let $x'_k = x_{n_k}$ and $y'_k = y_{n_k}$. Since $\sqrt{x}$ is unif cts. on $[0,1]$, if "$x'_k, y'_k \leq 1$" is frequently true, then, for some subseq. pair (x'_{k_i}) of (x'_k) and (y'_{k_i}) of (y'_k), we have $x'_{k_i}, y'_{k_i} \in [0,1]$ for all i, and, hence, $\left(\sqrt{x'_{k_i}} - \sqrt{y'_{k_i}}\right) \to 0$, which contradicts "$\forall k \; |\sqrt{x_k'} - \sqrt{y_k'}| \geq \varepsilon$."

Therefore, "$x_k', y_k' > 1$" is eventually true. Hence, $|\sqrt{x_k'} - \sqrt{y_k'}|$

$$= \left| \frac{(\sqrt{x_k'} - \sqrt{y_k'})(\sqrt{x_k'} + \sqrt{y_k'})}{\sqrt{x_k'} + \sqrt{y_k'}} \right| = \frac{|x_k' - y_k'|}{\sqrt{x_k'} + \sqrt{y_k'}}$$

$$\leq \frac{|x_k' - y_k'|}{\sqrt{1} + \sqrt{1}} = \frac{1}{2}|x_k' - y_k'| \text{ eventually.}$$

But $\frac{1}{2}|x_k' - y_k'| \to 0$. Therefore, $|\sqrt{x_k'} - \sqrt{y_k'}| < \varepsilon$ eventually, again contradicting "$\forall k \; |\sqrt{x_k'} - \sqrt{y_k'}| \geq \varepsilon$."

Therefore, "$(\sqrt{x_n} - \sqrt{y_n}) \not\to 0$" must be false. ∎