"Subsubsequence Lemma" (just a name I made up):

If *every* subsequence (a_{n_k}) of a given sequence (a_n) ~~...~~ has a subsequence $(a_{n_{k_i}})$ converging to a given real L, then (a_n) converges to L.

(Indirect) *Proof.* Suppose (a_n) & L are a counterexample. Then, for some $\varepsilon > 0$, $|a_n - L| \geq \varepsilon$ frequently. Hence, $|a_{n_k} - L| \geq \varepsilon$ for all k, (☆) for some subsequence (a_{n_k}). But (a_{n_k}) has a subseq. $(a_{n_{k_i}})$ converging to L. So, eventually $|a_{n_{k_i}} - L| < \varepsilon$, contradicting (☆). ∎

ε-free proof of Uniform Continuity Theorem:

Assume A compact & $f: A \to \mathbb{R}$ cts.

We will prove f is unif. cts.

Assume $x_n, y_n \in A$ & $(x_n - y_n) \to 0$.

We will prove $(f(x_n) - f(y_n)) \to 0$.

Let $(x'_k) = (x_{n_k})$ & $(y'_k) = (y_{n_k})$

for some $n_1 < n_2 < n_3 < n_4 < \cdots$.

By the Subsubsequence Lemma, it's enough to find $k_1 < k_2 < k_3 < k_4 < \cdots$ such that $(f(x'_{k_i}) - f(y'_{k_i})) \to 0$.

By compactness, some subseq. $(x''_i) = (x'_{k_i})$ of (x'_k) converges to some $c \in A$. Let $(y''_i) = (y'_{k_i})$.

Since $(x_n - y_n) \to 0$, its subseq's $(x'_k - y'_k)$ and $(x''_i - y''_i)$ also converge to 0. Hence,

$$(y''_i) = (x''_i - (x''_i - y''_i)) \to c - 0 = c.$$

By continuity of f, $(f(x''_i)) \to c$ and $(f(y''_i)) \to c$. Therefore,

$$(f(x''_i) - f(y''_i)) \to c - c = 0. \blacksquare$$