9-14

#4 Section 4

[* Suppose every nonempty $B \subset \{1, \cdots, n\}$ has a max]

Prove every nonempty $A \subset \{1, \cdots, n+1\}$ has a max.

(Eg $n=6$

$A = \{1, 3, 5, 6, 7\} \subset \{1, \cdots n+1\}$

$max(A) = 7 = n+1$)

If $E \subset \{1, \cdots, n+1\}$ and $n+1 \in E$, what is $max(E)$?

Everything in E is in $\{1, \cdots, n+1\}$ so every $x \in E$ is $x \leq n+1$

Case $n+1 \notin A$: $\emptyset \neq A \subset \underbrace{\{1, \cdots, n+1\} - \{n+1\}}_{\{1, \cdots, n\}}$ by our inductive

hypothesis * applied to A, A has a max

Today: §7

The set of finite nonempty binary strings is countable,

meaning there is an onto function (surjection) from $\mathbb{Z}_+$

$1 \to 0$	$5 \to 10$	$9 \to 010$	$13 \to 110$	
$2 \to 1$	$6 \to 11$	$10 \to 011$	$14 \to 111$	
$3 \to 00$	$7 \to 000$	$11 \to 100$	$15 \to 0000$	
$4 \to 01$	$8 \to 001$	$12 \to 101$	$16 \to 0001$	etc...

The set is also infinite (obviously), so we call it

countably infinite

Let A = the set of $f: \mathbb{Z}_+ \to \{0, 1\}$, the set of all

infinite binary sequences/strings.

Cantor's BIG Breakthrough

I claim A is uncountable, that is, if $g: \mathbb{Z}_+ \to A$, then g is not onto, i.e. for some $a \in A$ $a \neq g(n)$ for all $n \in \mathbb{Z}_+$

Write $g(1) = x_{11}\ x_{12}\ x_{13}\ x_{14}\ x_{15} \cdots x_{1n} \cdots$ where $x_{1n} = (g(1))(n) \in \{0,1\}$

$g(2) = x_{21}\ x_{22}\ x_{23}\ x_{24}\ x_{25} \cdots x_{2n} \cdots$

$g(3) = x_{31}\ x_{32}\ x_{33}\ x_{34}\ x_{35} \cdots x_{3n} \cdots$

$\vdots$

$g(k) = x_{k1}\ x_{k2}\ x_{k3}\ x_{k4}\ x_{k5} \cdots x_{kn} \cdots$

$\vdots$

Define $a \in A$ by $a(n) = 1 - x_{nn}$

$x_{11} = 0 \Rightarrow a(1) = 1 - 0 = 1$

$x_{11} = 1 \Rightarrow a(1) = 1 - 1 = 0$

$x_{55} = 0 \Rightarrow a(5) = 1 - 0 = 1$

$x_{55} = 1 \Rightarrow a(5) = 1 - 1 = 0$

For all $n \in \mathbb{Z}_+$, $a(n) = 1 - x_{nn} = 1 - g(n)(n)$, so $a(n) \neq g(n)(n)$

so $a \neq g(n)$ $\square$

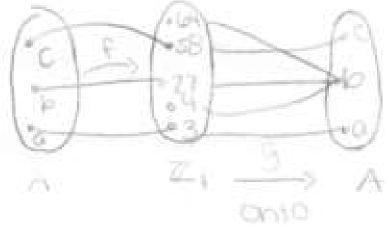
Theorem $\underset{\text{Read}}{7.8}$: For every set A, there is no onto $f: A \to P(A)$

Corollary: $P(\mathbb{Z}_+)$ is uncountable

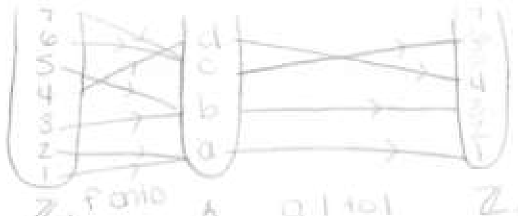
Corollary: There is no surjection from $P(\mathbb{Z}_+)$ to $P(P(\mathbb{Z}_+))$

The following are equivalent if $A \neq \emptyset$

- there is $f: \mathbb{Z}_+ \to A$ onto
- there is $f: A \to \mathbb{Z}_+$ 1-to-1
- Either A is finite or there is a bijection $f: A \to \mathbb{Z}_+$
- Either A is finite or there is a bijection $f: \mathbb{Z}_+ \to A$.



Picture of (2) ⇒ (1)



(1) ⇒ (2)

For all $x \in A$, $g(x)$ is $\min(f^{-1}(\{x\}))$

$$x \neq y \Rightarrow f^{-1}(\{x\}) \cap f^{-1}(\{y\})$$
$$= f^{-1}(\{x\} \cap \{y\})$$
$$= f^{-1}(\emptyset) = \emptyset$$
$$\min f^{-1}(\{x\}) \neq \min f^{-1}(\{y\})$$
$$g(\{x\}) \neq g(\{y\})$$

The following are equivalent

- A is ~~countabled infinite~~ countably infinite
- There is a bijection $f: A \to \mathbb{Z}_+$
- There is a bijection $f: \mathbb{Z}_+ \to A$

$g(n)$ is the unique $x \in A$ where $f(x) = n$, if such an a exists. Otherwise $g(n) = y$ for an arbitrary fixed $y \in A$

$\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+$ is countable

6	21						
5	15	20					
4	10	14	19				
3	6	9	13	18			
2	3	5	8	12	17		
1	1	2	4	7	11	16	22
	1	2	3	4	5	6	

A bijection from $\mathbb{Z}_+^2$ to $\mathbb{Z}_+$:

$$(x,y) \overset{F}{\longmapsto} \tfrac{1}{2}(x+y-2)(x+y-1)+y$$

$$\mathbb{Z}_+^3 \xrightarrow{\text{bijection}} \mathbb{Z}_+^2 \underset{\text{bij}}{\xrightarrow{F}} \mathbb{Z}_+$$

$$(a,b,c) \longmapsto (T(a,b),c)$$

In general, $\mathbb{Z}_+^n$ is countable

$\mathbb{Q} \cap (0,\infty)$ is countable:

Build h. $\mathbb{Z}_+ \xrightarrow{\text{onto}} \mathbb{Q}$

$$\mathbb{Z}_+ \xrightarrow{F^{-1}} \mathbb{Z}_+^2 \underset{\text{onto}}{\xrightarrow{g}} \mathbb{Q}$$

$g(m,n) = m/n$ if $n \neq 0$

$g(m,n) = 17$ if $n = 0$