9-16-10

last time we talked about countable & uncountable ∞

A nonempty set B is countable iff any of the following equivalents statements are true

1) There exists $f: \mathbb{Z}_+ \longrightarrow B$ onto (sur)

2) There exists $f: B \longrightarrow \mathbb{Z}_+$ 1-to-1 (inj)

3a) B is finite or there exists $f: B \rightarrow \mathbb{Z}_+$ 1-to-1 & onto (bij)

b) B is finite or there exists $f: \mathbb{Z}_+ \rightarrow B$ 1-to-1 & onto (bij)

last Time: $\mathbb{Q} \cap (0, \infty)$ is countable

$$\mathbb{Z}_+ \xrightarrow[\substack{\text{onto} \\ \text{1-to-1}}]{F^{-1}} \mathbb{Z}_+^2 \xrightarrow[\text{onto}]{g} \mathbb{Q} \cap (0, \infty)$$

$g(x,y) = x/y$

$F(x,y) = \frac{1}{2}(x+y-1)(x+y-2) + y$

$\hookrightarrow$ bijection from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to $\mathbb{Z}_+$

$F(1,4) = 10$

$F^{-1}(10) = (1,4)$

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10
6 9
3 5 8 12
1 2 4 7 11
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A_1, A_2 countable $\Rightarrow A_1 \cup A_2$ countable.

<u>Proof</u>: Let $f_i: \mathbb{Z}_+ \rightarrow A_i$ be onto for $i=1$ & $i=2$

Define $g: \mathbb{Z}_+ \longrightarrow A_1 \cup A_2$ by

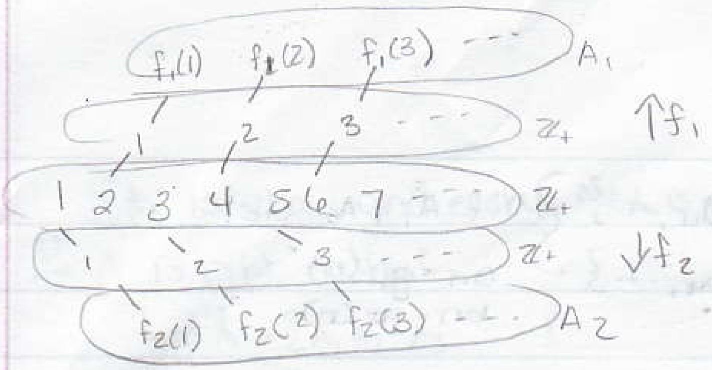
$g(2n) = f_1(n)$ and $g(2n-1) = f_2(n)$ for all $n \in \mathbb{Z}_+$

If $x \in A_1$, then $x = f_1(n)$ for some $n \in \mathbb{Z}_+$ so $x = g(2n)$

if $x \in A_2$, then $x = f_2(n)$ for some $n \in \mathbb{Z}_+$ so $y = g(2n-1)$

Thus, if $x \in A_1 \cup A_2$ then $x \in A$ or $x \in A_2$, so $x \in g(\mathbb{Z}_+)$

So, g is onto so $A_1 \cup A_2$ is countable



$A_1 = \{f_1(1), f_1(2), f_1(3), \ldots\}$

$A_2 = \{f_2(1), f_2(2), f_2(3), \ldots\}$

$A_1 \cup A_2 = \{\underbrace{f_2(1)}_{g_1}, \underbrace{f_1(1)}_{g_2}, \underbrace{f_2(2)}_{g_3}, \underbrace{f_1(2)}_{g_4}, \underbrace{f_2(3)}_{g_5} \ldots\}$

$A_1 \cup A_2 \cup A_3$

If A_1 A_2 A_3 are countable is $A_1 \cup A_2 \cup A_3$ countable?

$A_1 A_2$ are countable $\Rightarrow A_1 \cup A_2$ ctbl

$A_1 \cup A_2, A_3$ ctbl $\Rightarrow (A_1 \cup A_2) \cup A_3$ ctbl

In general $A_1 \cup \cdots \cup A_n$ is ctbl if $A_1, A_2, A_3, \ldots A_n$ are all countable.

If A_1 A_2 ctbl then $A_1 \times A_2$ is ctbl

$h(x,y) = (g_1(x), g_2(y))$

$$\begin{tikzcd} \mathbb{Z}_+ \arrow[r, "F^{-1}", "\text{1-to-1 onto}"'] & \mathbb{Z}_+ \arrow[d, "g_1"'] & \times\, \mathbb{Z}_+ \arrow[d, "g_2"] \arrow[r, "h"] & A_1 \times A_2 \\ & A_1 & A_2 & \end{tikzcd}$$

$g_1\ g_2$ onto

$h \circ F^{-1}$ is onto

$A_1 = \{a_1, a_2, a_3, a_4, \dots\}$ $\Big\}$ if A_1, A_2 ctbl

$A_2 = \{b_1, b_2, b_3, b_4, \dots\}$ $\quad a_n = g_1(n)$, $b_n = g_2(n)$

$$A_1 \times A_2 = \{(a_1, b_1), (a_2, b_2), (a_2, b_1), (a_1, b_3), (a_2 b_2) (a_3, b_1), (a_1, b_4), (a_2, b_3), (a_3, b_2), \underbrace{(a_4, b_1)}_{h(F^{-1}(10))} \dots\}$$

If A_1, A_2 ctbl, then $A_1 \times A_2$ is ctbl

If A_3 is ctbl too, then $A_1 \times A_2 \times A_3 = (A_1 \times A_2) \times A_3$ is ctbl.

$A_1, \dots, A_n$ ctbl $\Rightarrow \prod_{i=1}^{n} A_i$ ctbl

But $\{0,1\}$ is ctbl and $\prod_{t \in \mathbb{Z}_+} \{0,1\}$ is not ctbl

So in general,

(A_n ctbl for all $n \in \mathbb{Z}_+$) $\not\Rightarrow \prod_{n \in \mathbb{Z}_+} A_n$ ctbl

But, (A_n ctbl for al $n \in \mathbb{Z}_+$) $\Rightarrow \bigcup_{n \in \mathbb{Z}_+} A_n$ ctbl.

$A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, \dots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, \dots\}$

$A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, \dots\}$

$\vdots$

Let $g_n : \mathbb{Z}_+ \to A_n$ onto for all $n \in \mathbb{Z}_+$

$h(x,y) = g_x(y)$ for all $(x,y) \in \mathbb{Z}_+^2$

Then $h \circ f^{-1} : \mathbb{Z}_+ \to \bigcup_{n \in \mathbb{Z}_+} A_n$ is onto

Recall $F(x,y) = \frac{1}{2}(x+y-1)(x+y-2)+y$

If $\mathcal{A}$ is ctbl, and every $A \in \mathcal{A}$ is ctbl, then
$\bigcup_{A \in \mathcal{A}} A$ is ctbl. Why?

Let $\mathcal{A} = \{A_1, A_2, A_3, \dots\}$

$\bigcup_{A \in \mathcal{A}} A = \bigcup_{n \in \mathbb{Z}_+} A_n$ is ctbl.

$\mathbb{Q} \cap (0, \infty)$ is ctbl $\leftarrow$ so there is $g: \mathbb{Z}_+ \to \mathbb{Q} \cap (0, \infty)$ onto

There is a bijection $f: \mathbb{Q} \cap (0, \infty) \to \mathbb{Q}(-\infty, 0)$.

$f(q) = -q$ $\qquad f \circ g: \mathbb{Z}_+ \to (-\infty, 0)$ is onto

So, $\mathbb{Q} \cap (-\infty, 0)$ is ctbl

So $\mathbb{Q} = (\mathbb{Q} \cap (-\infty, 0)) \cup (\mathbb{Q} \cap (0, \infty))$ is ctbl.

For all $x \in (0, \infty)$ let $A_x = \{q + \sqrt{x} \mid q \in \mathbb{Q}\}$

$$\mathbb{Z}_+ \xrightarrow{\text{onto}} \mathbb{Q} \longrightarrow A_x$$
$$q \to q + \sqrt{x}$$

So A_x is ctbl

So $\bigcup_{r \in \mathbb{Q}} A_r$ is ctbl.

Thm 9.1 says Dedekind is equivalent to infinite is equivalent to having an injection from $\mathbb{Z}_+$

Thm 9.2 The axiom of choice implies that $\prod_{A \in \mathcal{A}} A$ is always nonempty, unless $\emptyset \in \mathcal{A}$.

$\prod_{A \in \mathcal{A}} A = \{f: \mathcal{A} \to \bigcup \mathcal{A} \mid f(A) \in A \text{ for all } A \in \mathcal{A}\}$

$= \{\text{choice functions for } \mathcal{A}\}$

 Axiom of choice.