Jose Arredondo

Notes 9-30-10

3.) $Y = [-1, 1]$ subspace of $\mathbb{R}$ (with the standard topology)

- $A = \{x \mid \frac{1}{2} < |x| < 1\}, = (-\frac{1}{2}, -1) \cup (\frac{1}{2}, 1)$
  A open in Y? Yes.
  A is open in $\mathbb{R}$, and $A = A \cap Y$. & $A \subset Y$

- $B = \{x \mid \frac{1}{2} < |x| \leq 1\}$ $[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$
  B open in Y? Yes
  B open in $\mathbb{R}$? No

Is there U open in $\mathbb{R}$ such that $U \cap Y = B$?

$U = (-17, -\frac{1}{2}) \cup (\frac{1}{2}, 43)$
$U \cap Y = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1] = B$

- $C = \{x \mid \frac{1}{2} \leq |x| < 1\}$, $(-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$

  C open in $\mathbb{R}$? No.
  C open in Y? No.

If U open $\subset \mathbb{R}$ $\ni$ $U \cap Y = C$, then $\frac{1}{2} \in C = U \cap Y \subset U$, so $\frac{1}{2} \in U$. But then we have $(p, q) \subset U$ where $p < \frac{1}{2} < q$.
Choose $r \in \mathbb{R}$ such that $\underbrace{p < r, -\frac{1}{2} < r}_{\max\{p, -\frac{1}{2}\} < r}$, and $r < \frac{1}{2}$.

Then $r \in (p, q) \subset U$ and $-1 < \underbrace{-\frac{1}{2} < r < \frac{1}{2}}_{r \notin C} < 1$, so $r \in [-1, 1] = Y$.
So, $\underbrace{r \in U \cap Y = C}_{r \in C}$. $\Rightarrow\Leftarrow$

10.) Let $I = [0,1]$. Compare the product topology on $I \times I$, the dictionary order topology $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

$I = [0,1]$ (gets subspace topology from $\mathbb{R}$

$T_P =$ product topology on I^2 $(= I \times I)$

$T_D =$ dictionary order top on I^2

$T_S =$ subspace top. from $\mathbb{R}^2$ w/ dict. order top.

$T_P \subset T_D$?

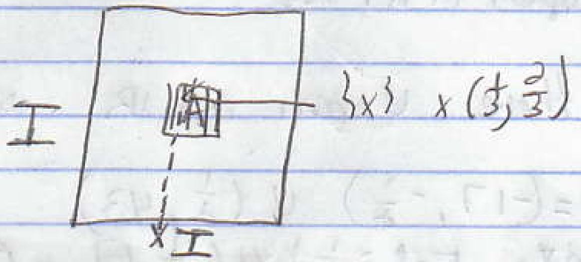
$T_D \subset T_P$?

$T_P \subset T_S$?

$T_S \subset T_P$?

$T_D \subset T_S$?

$T_S \subset T_D$?



$A = (\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3}) = \{(x,y) \mid \frac{1}{3} < x < \frac{2}{3} ; \frac{1}{3} < y < \frac{2}{3}\}$

$A \in T_P$ Yes $\qquad T_P = \{\bigcup \mathcal{E} \mid \forall E \in \mathcal{E}\ \exists U, V \text{ open} \subset I$
$A \in T_D$ Yes $\qquad E = U \times V\}$
$A \in T_S$ Yes $\qquad A = \bigcup \{(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})\}$

Is there U open $\subset \mathbb{R}^2$ w/ dict. order. top.
$U \cap I^2 = A$? Yes would $U = A$ work?

$(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3}) = \bigcup_{\frac{1}{3} < x < \frac{2}{3}} (\{x\} \times (\frac{1}{3}, \frac{2}{3}))$, $y = x$, $x \le y$, $y \le x$

$\{x\} \times (\frac{1}{3}, \frac{2}{3}) = \{(y,z) \in \mathbb{R}^2 \mid (x, \frac{1}{3}) < (y,z) < (x, \frac{2}{3})\}$

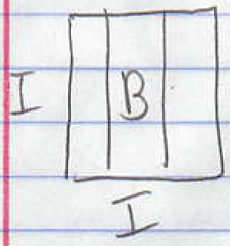
$((x, \frac{1}{3}), (x, \frac{2}{3}))$ $\qquad$ dictionary order

$\{(y,z) \in I^2 \mid (x, \frac{1}{3}) < (y,z) < (x, \frac{2}{3})\}$

Jose Arredondo

Notes. 9-30-10

$B=(\frac{1}{3},\frac{2}{3})\times\underbrace{[0,1]}_{I}$



$I = \mathbb{R}\cap I$ & $\mathbb{R}$ is open in $\mathbb{R}$, so I is open in I with respect to the subspace topology from $\mathbb{R}$.

$B\in T_p$: Yes

$B\in T_D$ Yes

$B\in T_S$ Yes

$(\frac{1}{3},\frac{2}{3})$ & I are open in I.

$C=[\frac{1}{3},\frac{2}{3}]\times[0,1]$

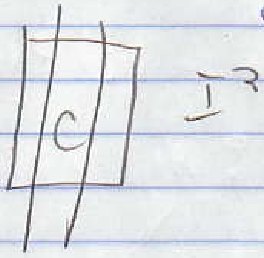
$C\notin T_p$

$C\notin T_D$

$C\in T_S$ Yes

$C= I^2\cap\underline{([\frac{1}{3},\frac{2}{3}]\times\mathbb{R})}$

open in $\mathbb{R}^2$ w/ dict. order.



$[1/3, 2/3] \times [0,1] \in \mathcal{T}_S \setminus (\mathcal{T}_D \cup \mathcal{T}_P)$

$\Rightarrow \mathcal{T}_S \not\subset \mathcal{T}_D$ & $\mathcal{T}_S \not\subset \mathcal{T}_P$

$[0,1] \times (1/2, 1] \in (\mathcal{T}_P \cap \mathcal{T}_S) \setminus \mathcal{T}_D$

$\Rightarrow \mathcal{T}_P \not\subset \mathcal{T}_D$ (and $\mathcal{T}_S \not\subset \mathcal{T}_D$, which we already knew)

$\left([0, 1/2) \times [0,1]\right) \cup \left(\{1/2\} \times [0, 1/3)\right)$

$\in (\mathcal{T}_D \cap \mathcal{T}_S) \setminus \mathcal{T}_P$

$\Rightarrow \mathcal{T}_D \not\subset \mathcal{T}_P$ (and $\mathcal{T}_S \not\subset \mathcal{T}_P$, which we knew)

One the other hand, $\mathcal{T}_P \subset \mathcal{T}_S$ & $\mathcal{T}_D \subset \mathcal{T}_S$

From solving Exercise #9

Why? If $U \in \mathcal{T}_P$, then $U = V \cap I^2$ where V is open in $\mathbb{R}^2$ with the product topology.

But the dictionary order topology on $\mathbb{R}^2$ is finer than the product topology on $\mathbb{R}^2$, so V is in the dict. ord. top. on $\mathbb{R}^2$, so $U \in \mathcal{T}_S$

~~Exercise: In general, if $X \supset Y$ & X has order topology, then the subsp.~~

~~$\mathcal{T}_D \supset \mathcal{T}_S$: By Lemma 13.3, we need only prove that if B is a basis open set in $\mathcal{T}_S$ (see the definition at the start of §16) and $(x,y) \in B$, then $(x,y) \in B' \subset B$ for some B',~~

$\mathcal{T}_D \subset \mathcal{T}_S$: Suppose $U \in \mathcal{T}_D$. Then U is a union of sets of form

$$L_{ab} = \{(x,y) \in I^2 \mid (0,0) \leq (x,y) < (a,b)\},$$
$$U_{ab} = \{(x,y) \in I^2 \mid (a,b) < (x,y) \leq (1,1)\}, \text{ or}$$
$$I_{abcd} = \{(x,y) \in I^2 \mid (a,b) < (x,y) < (c,d)\}$$

where a, b, c, d are in I. (See the definition of order topology in section §14.)

$$L_{ab} = (I \times [0,a)) \cup (\{a\} \times [0,b)) = I^2 \cap \underbrace{((\mathbb{R} \times (-\infty, a)) \cup (\{a\} \times (-\infty, b)))}_{\text{open in } \mathbb{R}^2 \text{ w/ dict. ord. top.}} \in \mathcal{T}_S$$

$$U_{ab} = (I \times (a, 1]) \cup (\{a\} \times (b, 1]) = I^2 \cap \underbrace{((\mathbb{R} \times (a, \infty)) \cup (\{a\} \times (b, \infty)))}_{\text{open in } \mathbb{R}^2 \text{ w/ dict. ord. top.}} \in \mathcal{T}_S$$

$\therefore I_{abcd} = L_{cd} \cap U_{ab} \in \mathcal{T}_S$ because every intersection of a finite subcollection of $\mathcal{T}_S$ is in $\mathcal{T}_S$.

$\therefore U \in \mathcal{T}_S$ because U is a union of sets in $\mathcal{T}_S$ and $\mathcal{T}_S$ is a topology.