10-5-10

"Proof" that $\overline{\bigcup_{\alpha\in J} A_\alpha} \subset \bigcup_{\alpha\in J}\overline{A_\alpha}$

Fine: $x\in \overline{\bigcup A_\alpha} \Rightarrow \forall U$ nbhd of x $U \cap \bigcup_{\alpha\in J} A_\alpha \neq \emptyset$

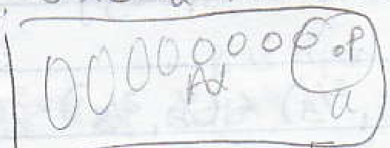
$\parallel$

$\bigcup_{\alpha\in J}(U\cap A_\alpha)$

$\Rightarrow U\cap A_\alpha\neq\emptyset$ for some α

$x\in\overline{A_\alpha}$ requires for one α $\emptyset\neq U\cap A_\alpha$ for all U nbhd of x

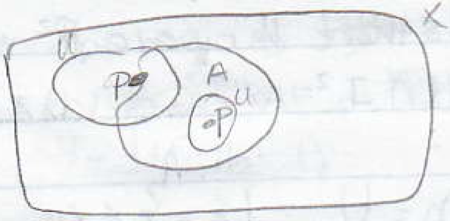
If $x\in\overline{\bigcup_{\alpha\in J}A_\alpha}$, then every nbhd U of x intersects some A_α, so $x\in\overline{A_\alpha}$ for some A_α. So, $x\in\bigcup_{\alpha\in J}\overline{A_\alpha}$.



Diff U's can hit diff A_α's

$\overline{A}=\{p\in X \mid$ for all U open $\subset X$, $p\in U\Rightarrow U\cap A\neq\emptyset\}$

$\overline{A}=\{p\in X\mid \forall(U$ nbhd of p$)\ U\cap A\neq\emptyset\}$



Concrete example

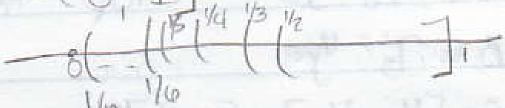
$X=\mathbb{R}$

$A_n=(\frac{1}{n},1]$

Thm 17.5b If B is a basis of X and $A\subset X$, then $\overline{A}=\{p\in X\mid \forall B\in\mathcal{B}\ (p\in B\Rightarrow A\cap B\neq\emptyset)\}$

$\overline{A_n}=[\frac{1}{n},1]$

$\bigcup_{n\in\mathbb{Z}_+}A_n=\bigcup_{n\in\mathbb{Z}_+}(1/n,1]=(0,1]$

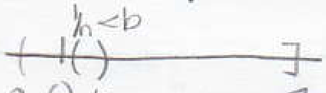


$\overline{\bigcup_{n\in\mathbb{Z}_+}A_n}=\overline{(0,1]}=[0,1]$

$\not\subset\bigcup_{n\in\mathbb{Z}_+}\overline{A_n}=\bigcup_{n\in\mathbb{Z}_+}[1/n,1]=(0,1]$

Consider $0 \in [0,1] = \overline{\bigcup_{n \in \mathbb{Z}^+} A_n}$

Every nbhd U of 0 intersects A_n for some n, but n depends on U.



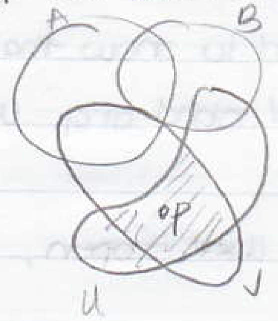
$(a,b) \cap (1/n, 1] = (1/n, b) \neq \emptyset$

Prove $\overline{A \cup B} \subset \bar{A} \cap \bar{B}$ ✿

We need to show $\underline{p \in \overline{A \cup B} \Rightarrow p \in \bar{A} \cup \bar{B}}$

Equivalently: $p \notin \bar{A} \cup \bar{B} \Rightarrow p \notin \overline{A \cup B}$

Suppose $p \notin \bar{A} \cup \bar{B}$. Then there are nbhd^s U & V of p st $U \cap A = \emptyset = V \cap B$.



U & V are open so $U \cap V$ is open, so $U \cap V$ is a nbhd of p & $(U \cap V) \cap (A \cup B) = \emptyset$ so $p \notin \overline{A \cup B}$.

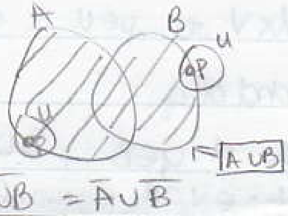
✿ $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$? Yes

Suppose $p \in \bar{A} \cup \bar{B}$. Then $p \in \bar{A}$ or $p \in \bar{B}$

<u>Case $p \in \bar{A}$</u>

Every nbhd U of p intersects A, so U intersects the bigger set $A \cup B$ too

Therefore $p \in \overline{A \cup B}$



<u>Case $p \in \bar{B}$</u> same argument.

✿ $\overline{A \cup B} = \bar{A} \cup \bar{B}$

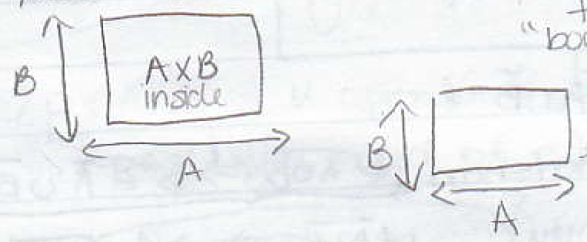
Prove $\{3\}$ is closed in $\mathbb{R}$.

$\mathbb{R}-\{3\} = (-\infty, 3) \cup (3, \infty)$

B is open so $\{3\}$ is closed.

Prove $\overline{A \times B} = \bar{A} \times \bar{B}$ $\overline{A\times B}$ is "inside" + "boundrary" $= \bar{A} \times \bar{B}$

Plausible



Prove ① $\bar{A} \times \bar{B} \subset \overline{A \times B}$

② $\overline{A\times B} \subset \bar{A} \times \bar{B}$ ✓

② Suppose $(p,q) \in \overline{A \times B}$. We need to show that $p \in \bar{A}$ & $q \in \bar{B}$

Suppose U nbhd of p & V nbhd of q. We need to show $U \cap A \neq 0$ & $V \cap B$

Since U is open & V is open, $U \times V$ is open, and $(p,q) \in U \times V$,

so $U \times V$ is a nbhd of (p,q)

So $(U \times V) \cap (A \times B) \neq \emptyset$, so pick $(r,s) \in (U \times V) \cap (A \times B)$

Then $r \in U \cap A$ & $s \in V \cap B$.

By hyp. $(p,q) \in \bar{A} \times \bar{B}$ we're deducing $(p,q) \in \overline{A \times B}$

① By Thm 17.5 b its enough to prove that every nbhd of (p,q) of the form $U \times V$ where U, V open is such that $(U \times V) \cap (A \times B) \neq \emptyset$

$(p,q) \in U \times V$ so $p \in U$ & $q \in V$ so U is a nbhd of p & V is a nbhd of q.

So $p \in \bar{A}$ & $q \in \bar{B}$, so $U \cap A \neq \emptyset$ & $V \cap B \neq \emptyset$ so if say $r \in U \cap A$ & $s \in V \cap B$ then $(r,s) \in (U \times V) \cap (A \times B)$