10-7-

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Notes

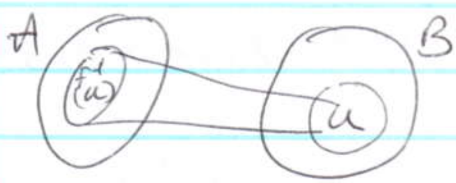
What is a homeomorphism?

A continous bijection with a continous inverse.

$f(x) = x$ is $\underbrace{\text{cts}}_{\text{continous}}$. from $\mathbb{R}$ to $\mathbb{R}$.

$$g(x) = \begin{cases} 1 : & x \geq 0 \\ 0 : & x < 0 \end{cases}$$ is not cts from $\mathbb{R}$ to $\mathbb{R}$.

Continous functions



u open $\Rightarrow f^{-1}(u)$ open

u open $\subset \mathbb{R} \Rightarrow f^{-1}(u) = u$ is open

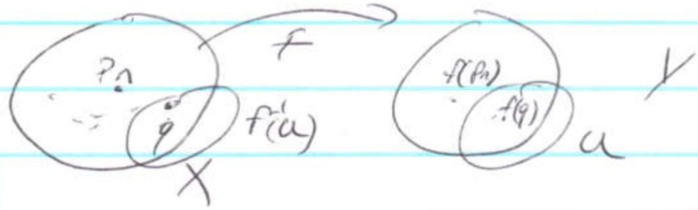
$g^{-1}((1,\infty)) = \{x \mid g(x) \in (1,\infty)\} = \emptyset$ is open

$g^{-1}((-1,2)) = \{x \mid g(x) \in (-1,2)\} = \mathbb{R}$ is open

$g^{-1}((-1,\tfrac{1}{2})) = \{x \mid g(x) \in (-1,\tfrac{1}{2})\} = (-\infty, 0)$ is open

$g^{-1}((\tfrac{1}{2},2)) = \{x \mid g(x) \in (\tfrac{1}{2},2)\} = [0,\infty)$ Not open

Fact: If a sequence $(p_n)_{n\in\mathbb{Z}_+}$ converges to q in X and $f: X \to Y$ is cts, then $(f(p_n))_{n\in\mathbb{Z}_+}$ converges to $f(q)$ in Y.



Pf. Let u be a nbhd of $f(q)$. Then $f^{-1}(u)$ is open & $q \in f^{-1}(u)$. So, $f^{-1}(u)$ is a nbhd of q. So all but finitely many p_n's are in $f^{-1}(u)$. ∴

$(1/n)_{n\in\mathbb{Z}_+}$ converges to 0 in $\mathbb{R}$.
If u is a nbhd. of 0, then $0 \in (a,b) \subset u$ for some a, b because $\{(a,b) \mid a<b\}$ is a basis of $\mathbb{R}$.

For all $n > 1/b$, we have $a < 0 < 1/n < b$, so $1/n \in (a,b) \subset u$.

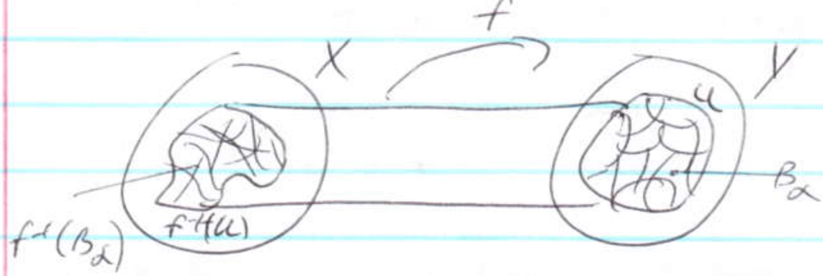
∴ All but finitely $f(p_n)$'s are in $f(f^{-1}(u)) = u$

$g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ is not cts.

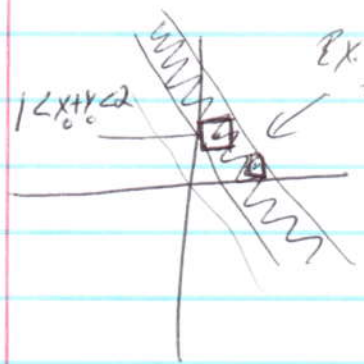
b/c $(-1/n)_{n\in\mathbb{Z}_+}$ converges to 0, but $(g(-1/n))_{n\in\mathbb{Z}_+} = (0)_{n\in\mathbb{Z}_+}$ does not converge to $g(0) = 1$

Prove $f: \mathbb{R}^2 \to \mathbb{R}$ is continous if $f(x,y) = x+y$.

$\{(a,b) \mid a<b\}$ is a basis for $\mathbb{R}$.



So, it's enough to prove that every $f^{-1}((a,b))$ is open.

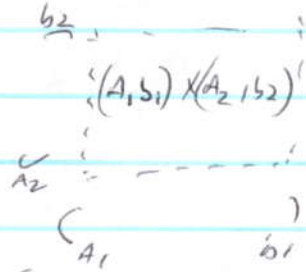


$\{\underbrace{(x,y)}_{\text{pair}} \mid x+y \in \underbrace{(a,b)}_{\text{interval}}\} = \{(x,y) \mid a<x+y<b\}$

Since $\{(a,b) \mid a<b\}$ is a basis of $\mathbb{R}$,

$\{\underbrace{(a_1,b_1)\times(a_2,b_2)}_{\text{rectangle}} \mid a_1<b_1 \ \& \ a_2<b_2\}$

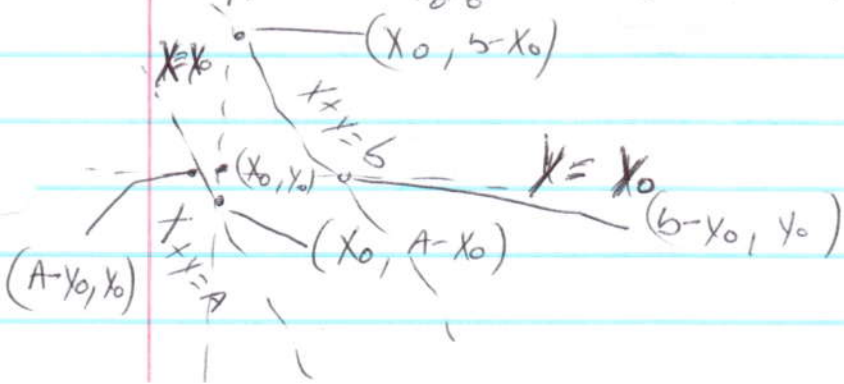
is a basis for $\mathbb{R}^2$



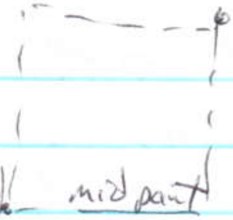
$\therefore$ $f^{-1}((a,b))$ is open if and only if, for all $(x,y) \in f^{-1}((a,b))$ $\exists$

$(a_1,b_1)\times(a_2,b_2)$ $(x,y) \in (a_1,b_1)\times(a_2,b_2) \subset f^{-1}((a,b))$.

Suppose $(x_0,y_0) \in f^{-1}((a,b))$. Then $a < x_0+y_0 < b$.



$(A - y_0, y_0)$



$\left(\underbrace{\frac{x_0+b-y_0}{2}}_{\cancel{A_2}\ b_1}, \underbrace{\frac{b-x_0+y_0}{2}}_{b_2}\right)$

$\left(\underbrace{\frac{A-y_0+x_0}{2}}_{A_1}, \underbrace{\frac{y+A-x_0}{2}}_{\cancel{B_2}\ A_2}\right)$ midpoint

$(x_0, A-x_0)$

I need to prove (1) $(x_0, y_0) \in (A_1, b_1) \times (A_2, \cancel{b}_2)$

(2) $(A_1, b_1) \times (A_2, b_2) \subset f^{-1}((A, b))$

(1) We're given $A < x_0 + y_0 < b$.

$A - y_0 < x_0 < b - y_0$

$$\frac{A-y_0}{2} < \frac{x_0}{2} < \frac{b-y_0}{2}$$

$$\underbrace{\frac{A-y_0}{2} + \frac{x_0}{2}}_{A_1} < \underbrace{\frac{x_0}{2} + \frac{x_0}{2}}_{x_0} < \underbrace{\frac{b-y_0}{2} + \frac{x_0}{2}}_{b_1}$$

So, $x_0 \in (A_1, b_1)$.

Similarly, $y_0 \in (A_2, b_2)$.

So, $(x_0, y_0) \in (A_1, b_1) \times (A_2, b_2)$.

(2) Suppose $(p, q) \in (A_1, b_1) \times (A_2, b_2)$

$A_1 < p < b_1$

$A_2 < q < b_2$

$\underbrace{A_1 + A_2}_{A} < p + q < \underbrace{b_1 + b_2}_{b}$

So, $(p, q) \in f^{-1}((A, b))$.

$\square$