10-26-10

distance ?

$(1,2)$ & $(4,4)$ in the square metric on $\mathbb{R}^2$?

$$\max\{|1-4|, |2-4|\} = \max\{3, 2\} = 3$$

$\vec{x}, \vec{y} \in \mathbb{R}^n$

Square: $\max_{1 \le i \le n} |x_i - y_i| = d(\vec{x}, \vec{y})$

euclidean: $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = d(\vec{x}, \vec{y})$

same for $n = 1$

uniform: $\max_{1 \le i \le n} (\min(|x_i - y_i|, 1))$

The ~~square & metric~~ square & euclidean metrics induce the same topology on $\mathbb{R}^n$ (for any choice of n)

Topologies measure infinite closeness of a point to a set. ($p \in \bar{A}$)

Metric measure this too ($0 = \inf\{d(p,q) \mid q \in A\}$)

Measure also measure non zero distance

$\mathcal{T}$ = product topology on $\mathbb{R}^2$ has basis

$\mathcal{T}_e$ = euclidean topology on $\mathbb{R}^2$ has basis

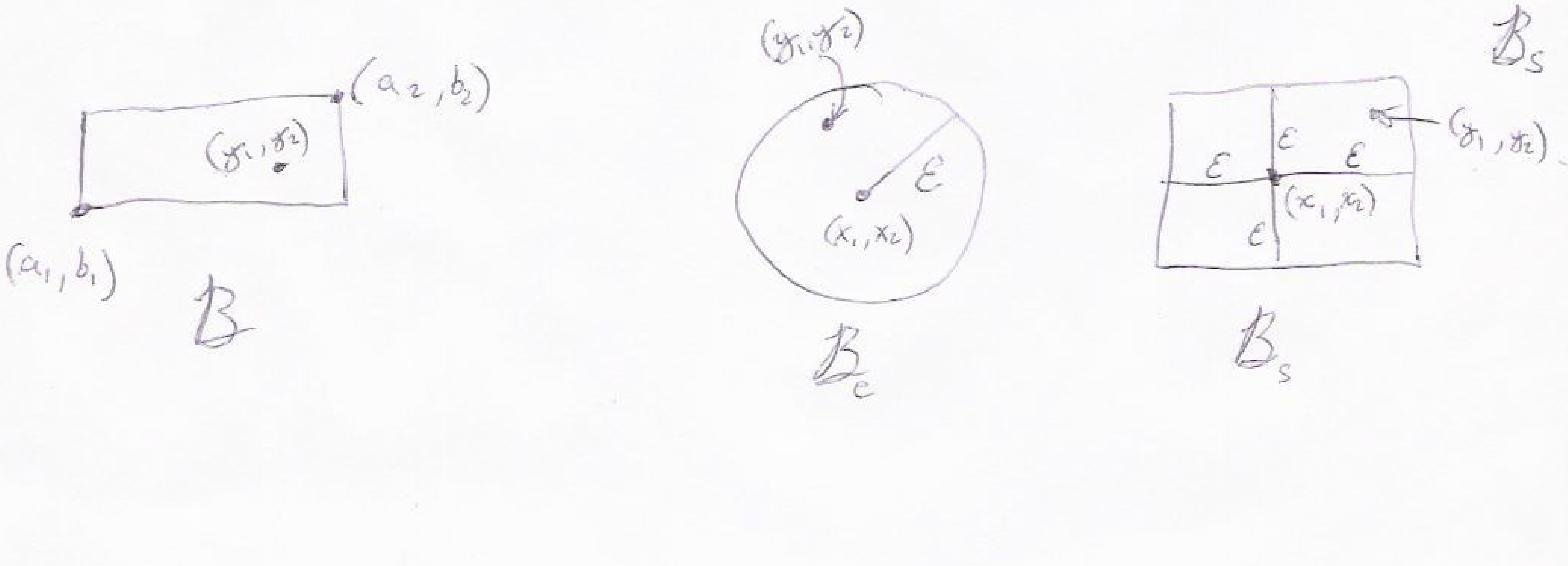
$\mathcal{T}_s$ = square topology on $\mathbb{R}^2$ has basis

$\cancel{\{(a_1 \times b_1)}$

$\{(a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\} = \mathcal{B}$

$\mathcal{B}_e = \{\{(y_1, y_2) \mid \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \varepsilon\} \mid x_1 x_2, \varepsilon \in \mathbb{R}, \varepsilon > 0\}$

$\{\{(y_1, y_2) \mid \max\{|x_1 - y_1|, |x_2 - y_2|\} < \varepsilon\} \mid \varepsilon, x_1, x_2 \in \mathbb{R}, \varepsilon > 0\} = \mathcal{B}_s$



I claim $\mathcal{T} = \mathcal{T}_e = \mathcal{T}_s$

Proof:

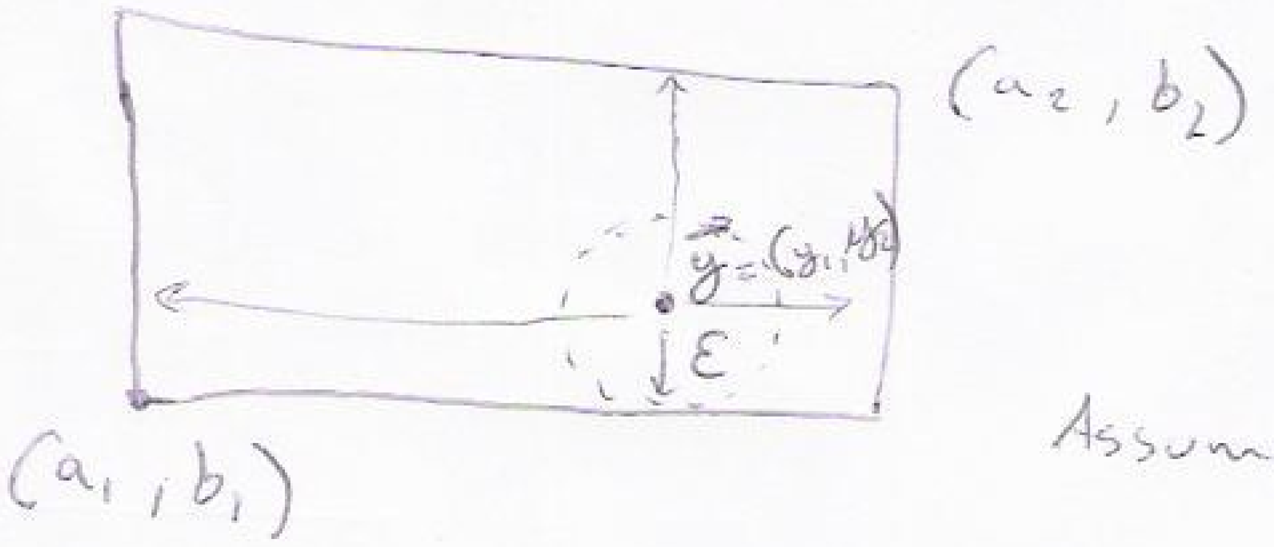
show $\mathcal{T} \subset \mathcal{T}_e \subset \mathcal{T}_s \subset \mathcal{T}$

$\mathcal{T} \subset \mathcal{T}_e$: Use Lemma 13.3

$\mathcal{T} \subset \mathcal{T}_e \iff \forall B \in \mathcal{B}\ \forall (y_1, y_2) \in B$

$\exists B_e \in \mathcal{B}_e\ (y_1, y_2) \in B_e \subset B$

$B = (a_1, b_1) \times (a_2, b_2)$



$\varepsilon = \min\{y_1 - a_1,\ a_2 - y_1,$
$b_2 - y_2,\ y_2 - b_1\}$

Assume $\sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2} < \varepsilon$

Prove $a_1 < z_1 < a_2$ & $b_1 < z_2 < b_2$

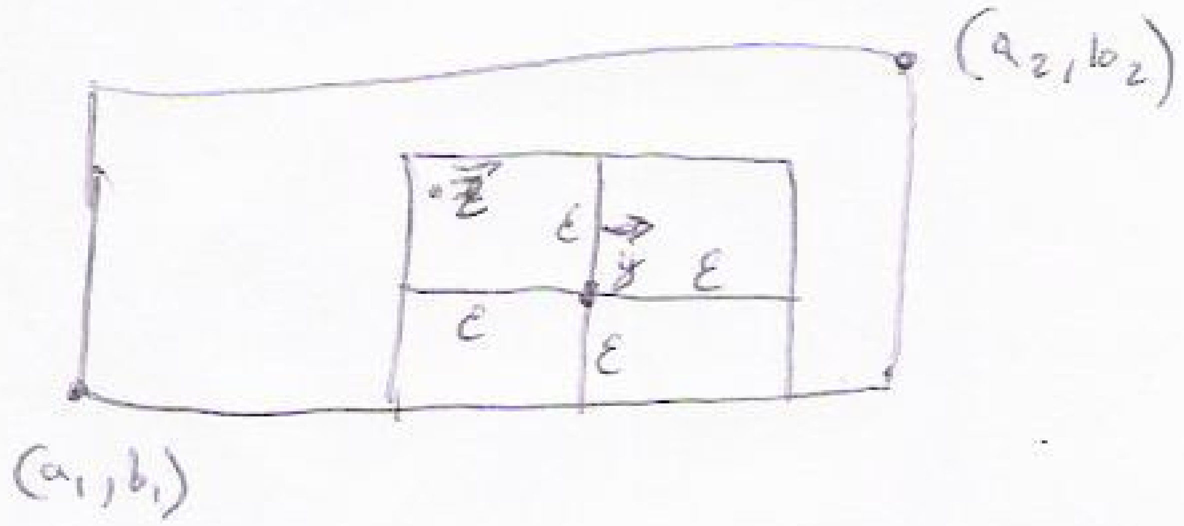
First step: $|z_1 - y_1|,\ |z_2 - y_2| < \varepsilon$

$(z_1 - y_1)^2 + (z_2 - y_2)^2 < \varepsilon^2$

$(z_1 - y_1)^2 < \varepsilon^2 - \underbrace{(z_2 - y_2)^2}_{\geq 0} \leq \varepsilon^2$

$\sqrt{(z_1 - y_1)^2} \leq \sqrt{\varepsilon^2}$

$|z_1 - y_1| \leq |\varepsilon| = \varepsilon$ similarly $|z_2 - y_2| < \varepsilon$



2nd step: Show square $\subset$ rectangle

$y_1 - a_1 \geq \varepsilon \Rightarrow y_1 - \varepsilon \geq a_1$

Similarly

$y_2 - \varepsilon \geq b_1$

$a_2 - y_1 \geq \varepsilon$

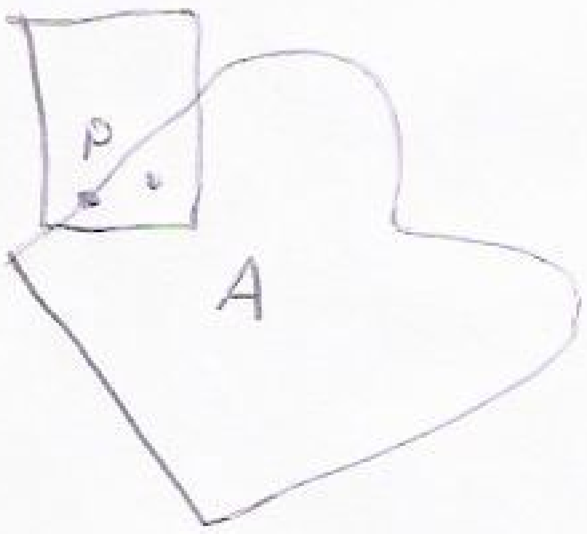
$a_2 \geq y_1 + \varepsilon$

Similarly, $b_2 \geq y_2 + \varepsilon$

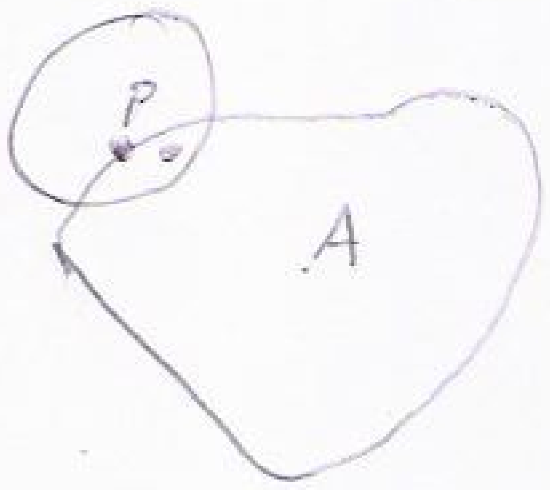
$$a_1 \le y_1 - \varepsilon < z_1 < y_1 + \varepsilon \le a_2$$

$$b_1 \le y_2 - \varepsilon < z_2 < y_2 + \varepsilon \le b_2$$

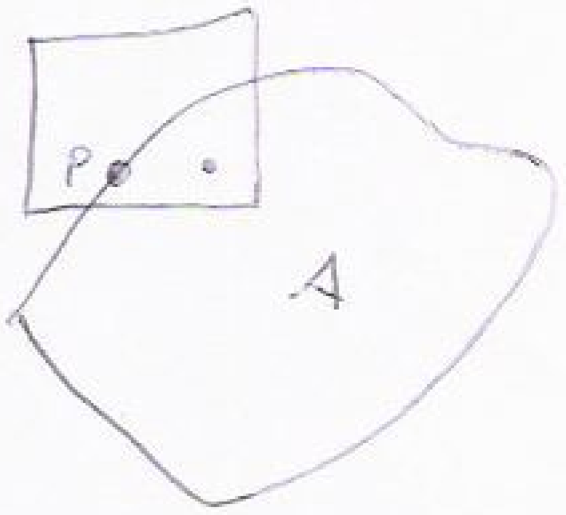
$$\begin{cases} a_1 < z_1 < a_2 \\ b_1 < z_2 < b_2 \end{cases} \qquad \text{QED}$$



$p \in \bar{A}$



$p \in \bar{A}$



$p \in \bar{A}$

$\mathcal{U}$ is open in topology generated by basis $\mathcal{B} \iff$

$\forall p \in \mathcal{U} \; \exists B \in \mathcal{B} \quad p \in B \subset \mathcal{U}$

Uniform metric on $\mathbb{R}^\omega$: $d(\vec{x}, \vec{y}) = \sup\{\bar{d}(x_n, y_n) \mid n \in \mathbb{Z}_+\} \leq \sup\{1 \mid n \in \mathbb{Z}\} = 1$

$\downarrow$

Includes uniform metric topology on $\mathbb{R}^\omega$

$\bar{d}(x_n, y_n) = \min\{1, \underbrace{d(x_n, y_n)}_{|x_n - y_n|}\} \leq 1$

For topology, only infinite closeness is "inherited" from a metric, so d & $\bar{d}$ induce the same topology.

$\vec{0} = (0, 0, 0, 0, \ldots) \in \mathbb{R}^\omega$

$\vec{f}_n = (\underbrace{0, 0, 0, \ldots, 0}_{n-1}, 1, 0, 0, 0, \ldots) \in \mathbb{R}^\omega$

$\vec{g}_n = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots) \in \mathbb{R}^\omega$

topology	Does $(\vec{f}_n)_{n \in \mathbb{Z}_+}$ converge to $\vec{0}$?	Does $(\vec{g}_n)_{n \in \mathbb{Z}_+}$ converges to $\vec{0}$?
product $\cap$	Yes	Yes
uniform metric $\cap$	no: $d(\vec{f}_n, \vec{0}) = 1$	Yes: $d(\vec{g}_n, \vec{0}) = \frac{1}{n}$
box product	no: $\vec{f}_n \notin \prod_{k \in \mathbb{Z}_+} (-1, 1) \ni \vec{0}$	no: $\vec{g}_n \notin \prod_{k \in \mathbb{Z}_+} (-\frac{1}{k}, \frac{1}{k}) \ni \vec{0}$