A sequential approach to uniform continuity and compact metric spaces

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Summary. We take some standard concepts and theorems about metric spaces that use "ε," prove the concepts equivalent to a sequential concepts that replace "ε" and any explicit inequalities with subsequences and limits, and prove the theorems using these subsequences.

- Continuity, uniform continuity, Cauchy sequences, totally bounded metric spaces, and compact metric spaces are characterized in terms of subsequences.
- Subsequences are used to prove that uniformly continuous images preserve the Cauchy property and total boundedness.
- Subsequences are used to prove that the compact metric spaces are exactly the complete totally bounded metric spaces.
- Subsequences are used to prove the Uniform Continuity Theorem.

Definition 1. Given a topological space X, $p \in X$, and a sequence $c_1, c_2, c_3, \ldots \in X$, we write $c_n \to p$ if for every neighborhood U of p there exists $N \in \mathbb{N}$ such that $c_N, c_{N+1}, c_{N+2}, \ldots \in U$.

Definition 2. Given a map between $f\colon X \to Y$ between metric spaces,

- f is *uniformly continuous* if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$;
- f is *sequentially uniformly continuous* if

$$d_X(a_n, b_n) \to 0 \text{ implies } d_Y(f(a_n), f(b_n)) \to 0.$$

Lemma 1. *A map between $f\colon X \to Y$ between metric spaces is uniformly continuous if and only if it is sequentially uniformly continuous.*

Proof. First, we assume uniform continuity and prove sequential uniform continuity. Given $d_X(a_n, b_n) \to 0$ and $\varepsilon > 0$, it is enough to find a tail of distances $(d_Y(f(a_n), f(b_n)))_{n \geq N}$ all $< \varepsilon$. Choose $\delta > 0$ such that, for all $a, b \in X$,

$d_X(a,b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$. Since $d_X(a_n, b_n) \to 0$, there is a tail of distances $(d_X(a_n, b_n))_{n \geq N}$ all $< \delta$. Therefore $d_Y(f(a_n), f(b_n)) < \varepsilon$ for all $n \geq N$.

Now we assume sequential uniform continuity and prove uniform continuity. Given $\varepsilon > 0$, it is enough to find $\delta > 0$ such that $d_X(a,b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$. Seeking a contradiction, suppose there is no such δ. For each $n \in \mathbb{N}$, case $\delta = \frac{1}{n}$ allows us to choose $a_n, b_n \in X$ such that $d_X(a_n, b_n) < \frac{1}{n}$ but $d_Y(f(a_n), f(b_n)) \geq \varepsilon$. Therefore, $d_X(a_n, b_n) \to 0$ but $d_Y(f(a_n), f(b_n)) \not\to 0$, in contradiction with sequential uniform continuity. □

Definition 3. We say that sequence $b_1, b_2, b_3, \dots$ is a *subsequence* of sequence $a_1, a_2, a_3, \dots$ if there exist $n_1 < n_2 < n_3 < \cdots$ such that $a_{n_k} = b_k$ for all $k \in \mathbb{N}$.

Definition 4. Suppose $a_1, a_2, a_3, \dots$ is a sequence in a metric space.

- Given $\varepsilon > 0$, we say $a_1, a_2, a_3, \dots$ is *ε-stable* if $d(a_m, a_n) < \varepsilon$ for all $m, n \in \mathbb{N}$.
- We say $a_1, a_2, a_3, \dots$ is *Cauchy* if, for every $\varepsilon > 0$, there is an ε-stable tail $a_N, a_{N+1}, a_{N+2}, \dots$.
- We say $a_1, a_2, a_3, \dots$ is *sequentially Cauchy* if $d(b_n, c_n) \to 0$ for all pairs of subsequences $b_1, b_2, b_3, \dots$ and $c_1, c_2, c_3, \dots$.

Lemma 2. *A sequence $a_1, a_2, a_3, \dots$ is Cauchy if and only if it is sequentially Cauchy.*

Proof. First, assume $a_1, a_2, a_3, \dots$ is Cauchy. Given subsequences $b_k = a_{m_k}$ and $c_k = a_{n_k}$, we will show that $d(b_k, c_k) \to 0$. To do this, we assume $\varepsilon > 0$ and then find a tail $d(b_K, c_K), d(b_{K+1}, c_{K+1}), d(b_{K+2}, c_{K+2}), \dots$ all less than ε. By assumption, there is an L such that $a_L, a_{L+1}, a_{L+2}, \dots$ is ε-stable. Choose K large enough that $m_K, n_K \geq L$. Then $d(b_k, c_k) < \varepsilon$ for all $k \geq K$. Thus, $a_1, a_2, a_3, \dots$ is sequentially Cauchy.

Now instead assume that $a_1, a_2, a_3, \dots$ is sequentially Cauchy. We will show it is Cauchy. Seeking a contradiction, suppose it is not, that $\varepsilon > 0$ and no tail $a_N, a_{N+1}, a_{N+2}, \dots$ is ε-stable. We will reach a contradiction by finding subsequences $p_1, p_2, p_3, \dots$ and $q_1, q_2, q_3, \dots$ such that $d(p_k, q_k) \geq \varepsilon$ for all $k \in \mathbb{N}$. Let $s_1 = t_1 = 1$ and then, given $n \in \mathbb{N}$ and given $s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n$, choose $t_{n+1} > s_{n+1} > t_n$ such that $d(a_{s_{n+1}}, a_{t_{n+1}}) \geq \varepsilon$, which is possible because the tail $a_{t_n+1}, a_{t_n+2}, a_{t_n+3}, \dots$ is not ε-stable. The resulting subsequences $p_k = a_{s_k}$ and $q_k = a_{t_k}$ are as desired. □

Theorem 1. *If $f \colon X \to Y$ is a uniformly continuous map between metric spaces and $a_1, a_2, a_3, \dots$ is Cauchy in X, then $f(a_1), f(a_2), f(a_3), \dots$ is Cauchy in Y.*

Proof. Given subsequences $p_1, p_2, p_3, \dots$ and $q_1, q_2, q_3, \dots$ of $f(a_1), f(a_2), f(a_3), \dots$, say, $p_k = f(a_{m_k})$ and $q_k = f(a_{n_k})$ where $m_1 < m_2 < m_3 < \cdots$ and $n_1 < n_2 < n_3 < \cdots$, let $b_k = a_{m_k}$ and $c_k = a_{n_k}$. Then $d(b_k, c_k) \to 0$ because $a_1, a_2, a_3, \dots$ is Cauchy. Since f is uniformly continuous, $d(p_k, q_k) = d(f(b_k), f(c_k)) \to 0$ too. □

Definition 5.

- A metric space X is *totally bounded* if, for each $\varepsilon > 0$, X has a finite open covering by ε-balls: $X = \bigcup_{i \leq n} B(x_i, \varepsilon)$.
- A metric space X is *sequentially totally bounded* if every sequence has a Cauchy subsequence.

Lemma 3. *A metric space X is totally bounded if and only if it is sequentially totally bounded.*

Proof. First we assume sequential total boundedness and prove total boundedness. Let $\varepsilon > 0$. Assuming that X is not covered by finitely many ε-balls, we will prove a contradiction. Define a sequence $x_1, x_2, x_3, \ldots$ by choosing $x_n \in X - \bigcup_{i<n} B(x_i, \varepsilon)$ for each $n \in \mathbb{N}$. Let $n_1 < n_2 < n_3 < \cdots$ be such that $y_k = x_{n_k}$ is a Cauchy subsequence. The sequence $y_1, y_2, y_3, \ldots$ has an ε-stable tail $y_K, y_{K+1}, y_{K+2}, \ldots$; hence, $y_{K+1} \in B(y_K, \varepsilon)$, which contradicts $x_{n_{K+1}} \in X - \bigcup_{i<n_{K+1}} B(x_i, \varepsilon)$.

Now we assume total boundedness and prove sequential total boundedness. Suppose $a_1, a_2, a_3, \ldots \in X$. For each $n \in \mathbb{N}$, let $F_n \subset X$ be finite and such that $X = \bigcup_{x \in F_n} B\left(x, \frac{1}{n}\right)$. Construct a chain $I_1 \supset I_2 \supset I_3 \supset \cdots$ of infinite subsets of $\mathbb{N}$ as follows. Let $I_0 = \mathbb{N}$. Given infinite $I_{n-1} \subset \mathbb{N}$ infinite, choose, for each $i \in I_{n-1}$, a point $f(i) \in F_n$ such that $a_{f(i)} \in B\left(f(i), \frac{1}{n}\right)$. Since I_{n-1} is infinite and F_n is finite, f must send infinitely many inputs to the same output. Choose $x_n \in F_n$ and $I_n = f^{-1}(\{x_n\})$ such that I_n is infinite. Then $I_{n-1} \supset I_n$ as required.

Now choose $n_1 < n_2 < n_2 < \cdots$ as follows. Given $m \in \mathbb{N}$ and n_i for all $i < m$, choose n_m from the infinite set I_m such that $n_m > n_i$ for all $i < m$. We claim that the resulting subsequence $b_m = a_{n_m}$ is Cauchy. To prove this, we will find, given $\varepsilon > 0$, an ε-stable tail $b_M, b_{M+1}, b_{M+2}, \ldots$. Choose M such that $\frac{1}{M} \leq \frac{\varepsilon}{2}$. Suppose $M \leq i, j$. Then $n_i, n_j \in I_M$. Hence, $d(b_i, b_j) \leq d(b_i, x_M) + d(x_M, b_j) < \varepsilon$. Thus, $b_M, b_{M+1}, b_{M+2}, \ldots$ is ε-stable. □

Theorem 2. *If $f\colon X \to Y$ is uniformly continuous and X is totally bounded, then $f(X)$ is totally bounded.*

Proof. Assume $f\colon X \to Y$ is uniformly continuous and X is totally bounded. Let $y_1, y_2, y_3 \ldots \in f(X)$ and choose x_n such that $y_n = f(x_n)$, for each $n \in \mathbb{N}$. By assumption, there are $n_1 < n_2 < n_3 < \cdots$ such that $w_1, w_2, w_3, \ldots$ is Cauchy where $w_m = x_{n_m}$. By Theorem 1, setting $z_m = f(w_m)$ makes $z_1, z_2, z_3, \ldots$ a Cauchy subsequence of $y_1, y_2, y_3 \ldots$. Thus, $f(X)$ is totally bounded. □

Lemma 4. *A convergent sequence in a metric space is Cauchy.*

Proof. Given $a_n \to x$ in a metric space and subsequences b_k and c_k, we have $b_k \to x$ and $c_k \to x$, which together imply $d(b_k, c_k) \leq d(b_k, x) + d(x, c_k) \to 0$. □

Lemma 5. *If $a_1, a_2, a_3, \ldots$ is a Cauchy sequence with convergent subsequence $b_k = a_{n_k} \to x$, then $a_n \to x$.*

Proof. Since $a_1, a_2, a_3, \ldots$ is sequentially Cauchy, $d(a_k, b_k) \to 0$. Therefore, $a_k \to x$. $\square$

Definition 6.

- A metric space X is *complete* if every Cauchy sequence converges.
- A topological space is *compact* if every open cover has a finite subcover.
- A metric space is *sequentially compact* if every sequence has a convergent subsequence.

Lemma 6. *A metric space X is sequentially compact if and only if it is sequentially totally bounded and complete.*

Proof. If X is sequentially totally bounded and complete, then every sequence has a Cauchy subsequence which converges, making X sequentially compact. To prove the converse, suppose that X is sequentially compact. Then every sequence has a convergent subsequence which is Cauchy, making X totally bounded. To see that X is also complete, suppose that $a_1, a_2, a_3, \ldots$ is Cauchy. By assumption, there is a convergent subsequence $b_m = a_{n_m} \to x$. By Lemma 5, $a_1, a_2, a_3, \ldots$ converges. $\square$

Lemma 7. *A metric space X is sequentially compact if and only if it is compact.*

Proof. First, suppose X is compact. Then X is totally bounded because if $\varepsilon > 0$ then $\{B(x, \varepsilon) \mid x \in X\}$ is an open cover with a finite subcover. Therefore, to show that X is sequentially compact, it suffices to show that X is complete. So, given $a_1, a_2, a_3, \ldots$ Cauchy, we will show that $a_n \to p$ for some $p \in X$. Let $C = \bigcap_{N \in \mathbb{N}} C_N$ where $C_N = \overline{\{a_n \mid n \geq N\}}$. Since each C_N is closed and nonempty and $C_1 \supset C_2 \supset C_3 \supset \cdots$, compactness of X implies that C is nonempty. Choose $p \in C$. Given $\varepsilon > 0$, choose N such that $a_N, a_{N+1}, a_{N+2}, \ldots$ is $\varepsilon/2$-stable. Since $p \in C \subset C_N$, the neighborhood $B(p, \varepsilon/2)$ of p intersects $\{a_n \mid n \geq N\}$. So, choose $M \geq N$ such that $d(p, a_M) < \varepsilon/2$. Then $d(a_n, p) \leq d(a_n, a_M) + d(a_M, p) < \varepsilon$ for all $n \geq N$. Thus, $a_n \to p$.

Now instead suppose that X is sequentially compact. Given an open cover $\mathcal{U}$ of X without a finite subcover, we will prove a contradiction. For each $n \in \mathbb{N}$, let $F_n \subset X$ be finite and such that $X = \bigcup_{x \in F_n} B\left(x, \frac{1}{n}\right)$. For each n, if $B\left(x, \frac{1}{n}\right)$ were covered by a finite $\mathcal{G}_x \subset \mathcal{U}$ for each $x \in F_n$, then $\bigcup_{x \in F_n} \mathcal{G}_x$ would be a finite subcover of $\mathcal{U}$. So, choose $x_n \in F_n$ such that $B\left(x_n, \frac{1}{n}\right)$ is not covered by any finite $\mathcal{G} \subset \mathcal{U}$. By sequential compactness, there are z and $n_1 < n_2 < n_3 < \cdots$ such that $y_k \to z$ where $y_k = x_{n_k}$. Then $B(z, \delta) \subset U \in \mathcal{U}$ for some δ, U. Choosing k large enough that $\frac{1}{n_k} \leq \frac{\delta}{2}$ and $d(y_k, z) \leq \frac{\delta}{2}$, we have

$$B\left(y_k, \frac{1}{n_k}\right) \subset B(z, \delta) \subset U \in \mathcal{U},$$

which contradicts $B\left(x_{n_k}, \frac{1}{n_k}\right)$ not being covered by any finite $\mathcal{G} \subset \mathcal{U}$. $\square$

Theorem 3. *A metric space X is compact if and only if it is totally bounded and complete.*

Proof. Combine Lemmas 6 and 7. $\square$

Lemma 8 (Subsubsequence Lemma)**.** *Given a topological space X, $p \in X$, and a sequence $c_1, c_2, c_3, \ldots \in X$, suppose that, for every subsequence $b_m = c_{n_m}$ where $n_1 < n_2 < n_3 < \cdots$, there exists a subsubsequence $a_k = b_{m_k}$ where $m_1 < m_2 < m_3 < \cdots$ such that $a_k \to p$. Then $c_n \to p$.*

Proof. Seeking a contradiction, suppose that $c_n \not\to p$. Then there is a neighborhood U of p such that U does not contain any tail $c_N, c_{N+1}, c_{N+2}, \ldots$. Choose $n_1 < n_2 < n_3 < \cdots$ by the following method of selecting n_m for each $m \in \mathbb{N}$. Given $m \in \mathbb{N}$ and n_i for $i < m$, choose $N > n_i$ for all $i < m$, observe that $\{c_N, c_{N+1}, c_{N+2}, \ldots\} \not\subset U$, and then choose $n_m \geq N$ such that $c_{n_m} \notin U$.

By construction, the subsequence $b_1, b_2, b_3, \ldots$ where $b_m = c_{n_m}$ avoids U. By assumption, $b_1, b_2, b_3, \ldots$ has a subsequence $a_k = b_{m_k} \to p$. Hence, some tail $a_K, a_{K+1}, a_{K+2}, \ldots$ is in U. But $a_K = b_{m_K} \notin U$ by construction. Contradiction! $\square$

Definition 7. Given a function $f\colon X \to Y$ between topological spaces,

- f is *continuous* if $p \in \overline{A}$ implies $f(p) \in \overline{f(A)}$;
- f is *sequentially continuous* if $a_n \to p$ implies $f(a_n) \to f(p)$.

Lemma 9. *Given a function $f\colon X \to Y$ between metric spaces, f is continuous if and only if it is sequentially continuous.*

Proof. Suppose f is sequentially continuous. Given $p \in \overline{A} \subset X$, choose $a_n \in A \cap B\left(p, \frac{1}{n}\right)$ for each $n \in \mathbb{N}$. Then $a_n \to p$. Hence, $f(a_n) \to f(p)$. Hence, $f(p) \in \overline{\{f(a_n) \mid n \in \mathbb{N}\}} \subset \overline{f(A)}$.

Now suppose f is continuous. Given $a_n \to p \in X$, we will show that $f(a_n) \to f(p)$. Let $b_m = a_{n_m}$ where $n_1 < n_2 < n_3 < \cdots$. By the Subsubsequence Lemma, it is enough to find $m_1 < m_2 < m_3 < \cdots$ such that $f(c_k) \to f(p)$ where $c_k = b_{m_k}$. Choose $m_1, m_2, m_3, \ldots$ as follows. Given $k \in \mathbb{N}$ and m_i for $i < k$, let $S = \{b_m \mid m_i < m \text{ for all } i < k\}$. Since every neighborhood of p contains a tail $a_N, a_{N+1}, a_{N+2}, \ldots$, every neighborhood of p intersects S. Therefore, $p \in \overline{S}$. Hence, $f(p) \in \overline{f(S)}$; hence, $f(S) \cap B\left(f(p), \frac{1}{k}\right) \neq \varnothing$. Therefore, we may choose m_k such that $d(f(b_{m_k}), f(p)) < \frac{1}{k}$ and such that $m_k > m_i$ for all $i < k$. Therefore, $d(f(c_k), f(p)) < \frac{1}{k} \to 0$, which implies $f(c_k) \to f(p)$. $\square$

Theorem 4. *If $f\colon X \to Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.*

Proof. Given $f\colon X \to Y$ continuous and $d_X(a_n, b_n) \to 0$, we will show that $d_Y(f(a_n), f(b_n)) \to 0$. Let $p_m = a_{n_m}$ and $q_m = b_{n_m}$ where $n_1 < n_2 < n_3 < \cdots$. By the Subsubsequence Lemma, it is enough to find $m_1 < m_2 < m_3 < \cdots$ such that $d_Y(f(r_k), f(s_k)) \to 0$ where $r_k = p_{m_k}$ and $s_k = q_{m_k}$. Since X

is compact, we may choose $x \in X$ and $m_1 < m_2 < m_3 < \cdots$ such that $r_k \to x$. Since $d_X(a_n, b_n) \to 0$, we also have $d_X(r_k, s_k) \to 0$. Therefore, $s_k \to x$. Since f is continuous, $f(r_k) \to f(x)$ and $f(s_k) \to f(x)$. Therefore, $d_Y(f(r_k), f(s_k)) \to 0$. $\square$