Discrete probability distributions

$$p_1, \dots, p_n \geq 0$$

$$p_1 + \dots + p_n = 1$$

Fair die:

$$p_1 = p_2 = p_3 = \dots = p_6 = \frac{1}{6}$$

Infinite example:

$$p_1 = \frac{1}{2}, \quad p_2 = \frac{1}{4}, \quad p_3 = \frac{1}{8}, \dots$$

$$\dots, p_k = 2^{-k}, \dots$$

I'll post notes to dkmj.org.

Continuous probability distributions

probability density function:

(PDF) $f(x) \geq 0$.

CDF = cumulative ~~density~~ distribution function:

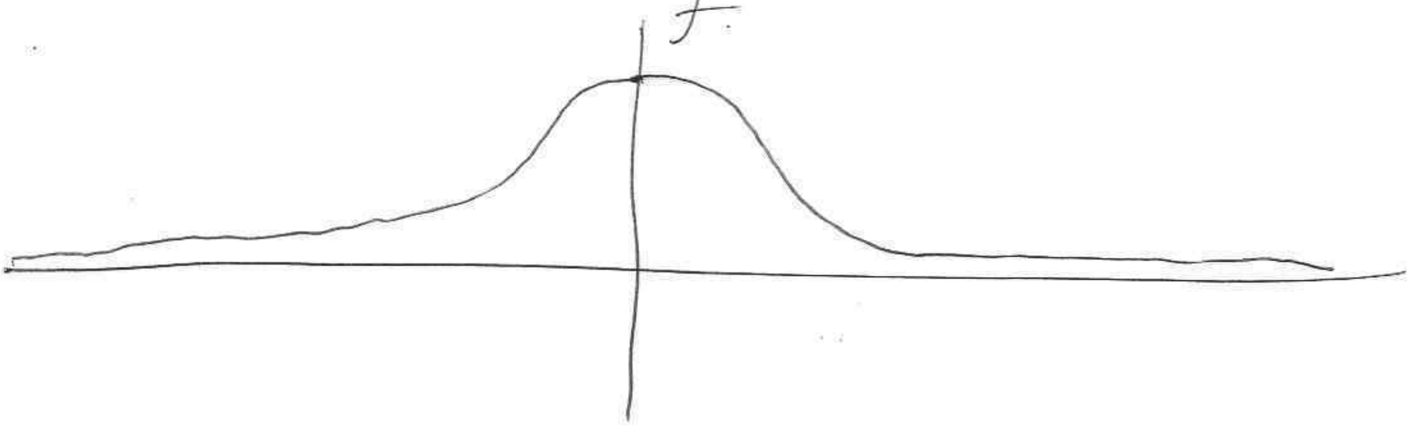
$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$F'(x) = f(x)$$

We require $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Bell curve / standard normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$\cancel{F(x) = \int_{-\infty}^{x} f(x)\,dx =}$ ...

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$\operatorname{erf}(x) = \int_{-x}^{x} e^{-t^2/2}\,dt/\sqrt{\pi}$$

$$F(x) = \int_{-\infty}^{x} e^{-t^2/2}\,dt/\sqrt{2\pi}$$

Interpretation of PDF & CDF

For any continuous distribution:

$$\int_a^b f(x)\,dx = P(a \le x \le b)$$

$$F(a) = P(x \le a)$$

(I'm assuming f is continuous.)

f = PDF = probability/length

For n fair coin tosses,

with n large, and

heads scored as $+1$;

tails scored as -1:

$$Z = \frac{\text{total score}}{\sqrt{n}}$$

$Z \sim N(0;1)$ (approx.)

This means Z is distributed according to the standard normal:

$$P(a \leq Z \leq b) = \int_a^b \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt \text{ (approx.)}$$

Example data: 60 heads

40 tails

$n = 100$

$$Z = \frac{60 - 40}{\sqrt{100}} = 2$$

$$P(|Z| \geq 2) = P(Z \leq -2 \text{ or } Z \geq 2)$$

$$= \int_{-\infty}^{-2} f(t)\,dt + \int_{2}^{\infty} f(t)\,dt$$

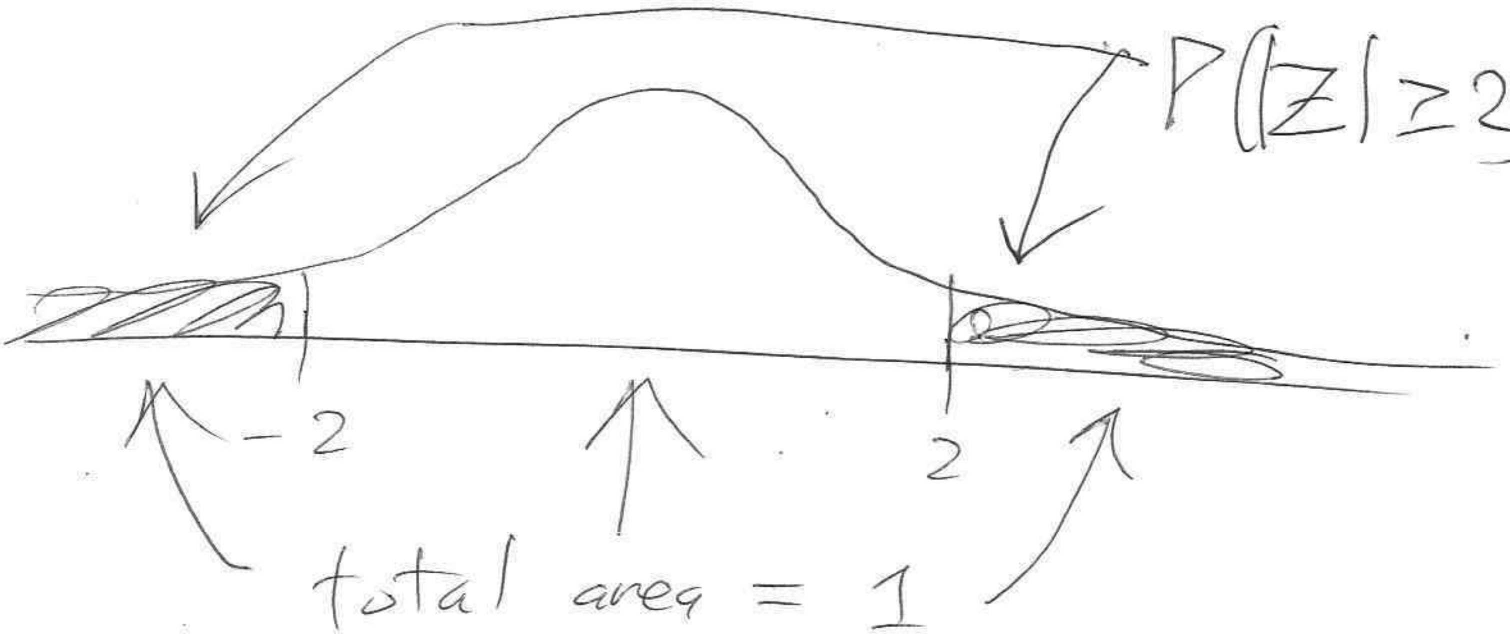
$$= 2\int_{2}^{\infty} f(t)\,dt$$ since

$f(t) = f(-t)$ where

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$\int_{-\infty}^{\infty} f(t)\,dt = 1$$

$$\int_{-\infty}^{-2} + \int_{-2}^{2} + \int_{2}^{\infty} = \int_{-\infty}^{\infty}$$



$$P(|Z| \geq 2) = 1 - \underbrace{\int_{-2}^{2} f(t)\,dt}_{\approx 0.954} \approx 4.6\%$$

$$\int_{-a}^{a} f(t)\,dt = \operatorname{erf}\left(\frac{a}{\sqrt{2}}\right)$$

(for standard normal)

coin: H T

$p_H = \frac{1}{2}$ $p_T = \frac{1}{2}$

die: 1 2 3 4 5 6

$p'_1 = p'_2 = \cdots = p'_6 = 1/6$

(coin, die)

H1	H2	H3	H4	H5	H6
T1	T2	T3	T4	T5	T6

Joint probability distribution

$q_{H1} = \frac{1}{12}$, $q_{H2} = \frac{1}{12}$, $q_{H3} = \frac{1}{12}$, ...

Independence means the probabilities multiply:

$$\frac{1}{12} = q_{T4} = p_T \cdot p'_4 = \frac{1}{2} \cdot \frac{1}{6}$$

More generally, events A & B are *independent* if

$$P(A \text{ and } B) = P(A)\,P(B).$$

A random variable is a function from experiment outcomes to the reals.

Every RV X has a

CDF: $F_X(b) = P(X \leq b)$

& PDF: $f_X(b) = F_X'(b)$,

excepting F discontinuous, in which case you want to consider discrete probabilities $P(X=b)$.

E.g., Z from coin toss is RV.

Z^2 is also an RV:

$$P(Z^2 \geq b) = 1 - P(Z^2 \leq b)$$

$$= 1 - P(-\sqrt{b} \leq Z \leq \sqrt{b})$$

$$= 1 - \int_{-\sqrt{b}}^{\sqrt{b}} e^{-t^2/2} \, dt / \sqrt{2\pi}$$

$$= 1 - \mathrm{erf}(\sqrt{b}/\sqrt{2})$$

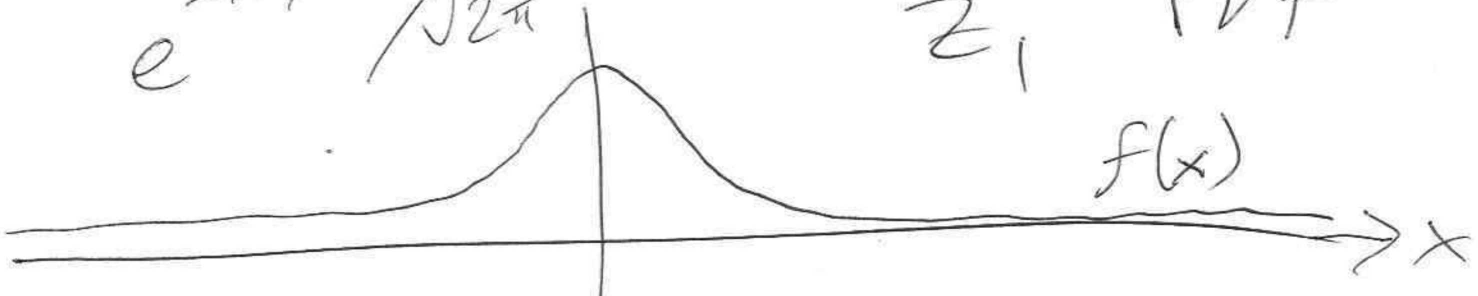
Z^2 has, by definition, the χ^2 distribution for "1 degree of freedom"

If $Z_1, \dots, Z_n$ are independent RV's each with standard normal distribution: $Z_i \sim N(0;1)$,

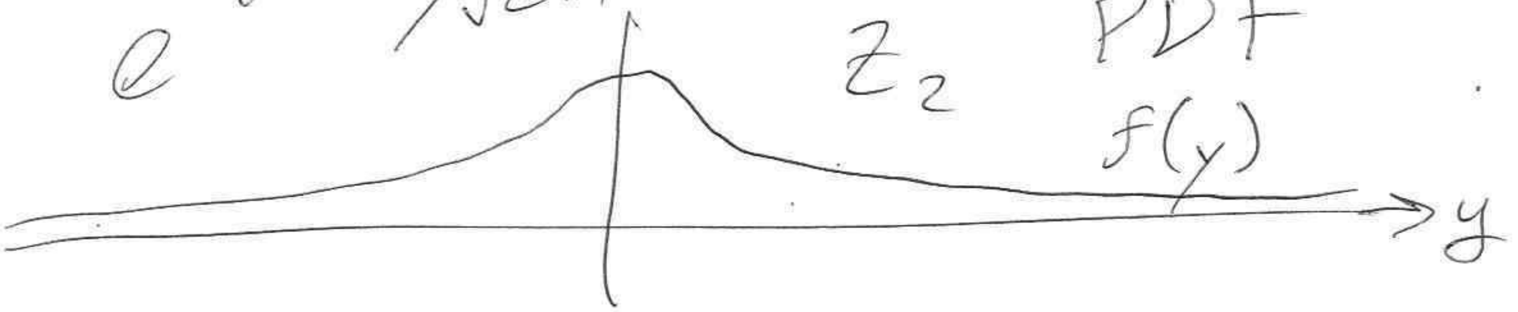
then we call the distribution of $Z_1^2 + \dots + Z_n^2$ ~~is called~~ the χ^2 distribution with n degrees of freedom.

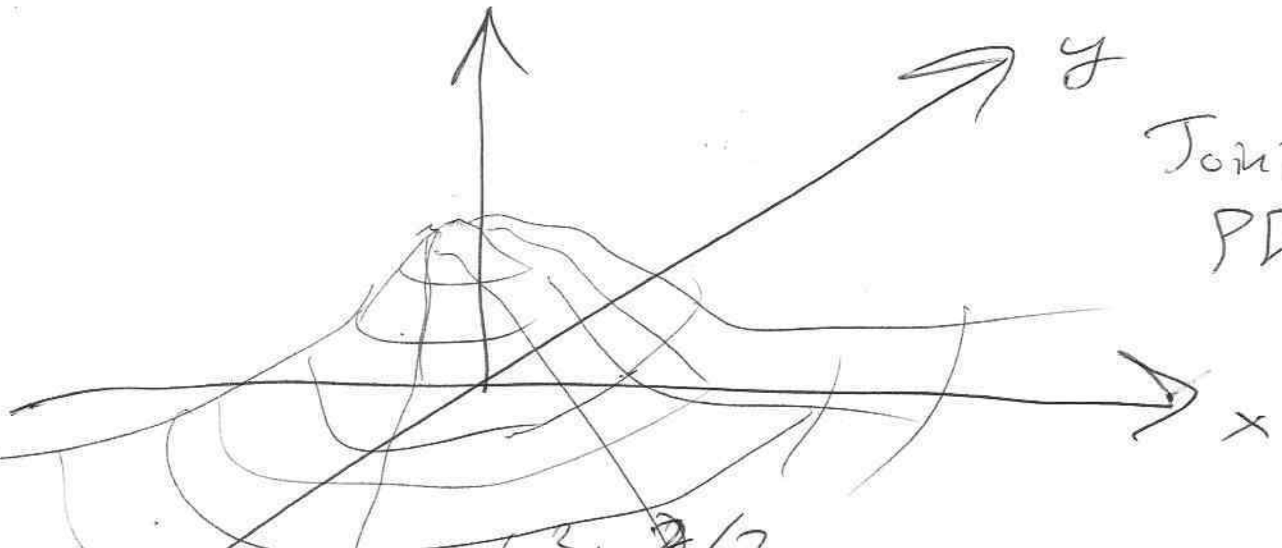
Case $n=2$:

$e^{-x^2/2}/\sqrt{2\pi}$ $\quad Z_1$ PDF



$e^{-y^2/2}/\sqrt{2\pi}$ $\quad Z_2$ PDF

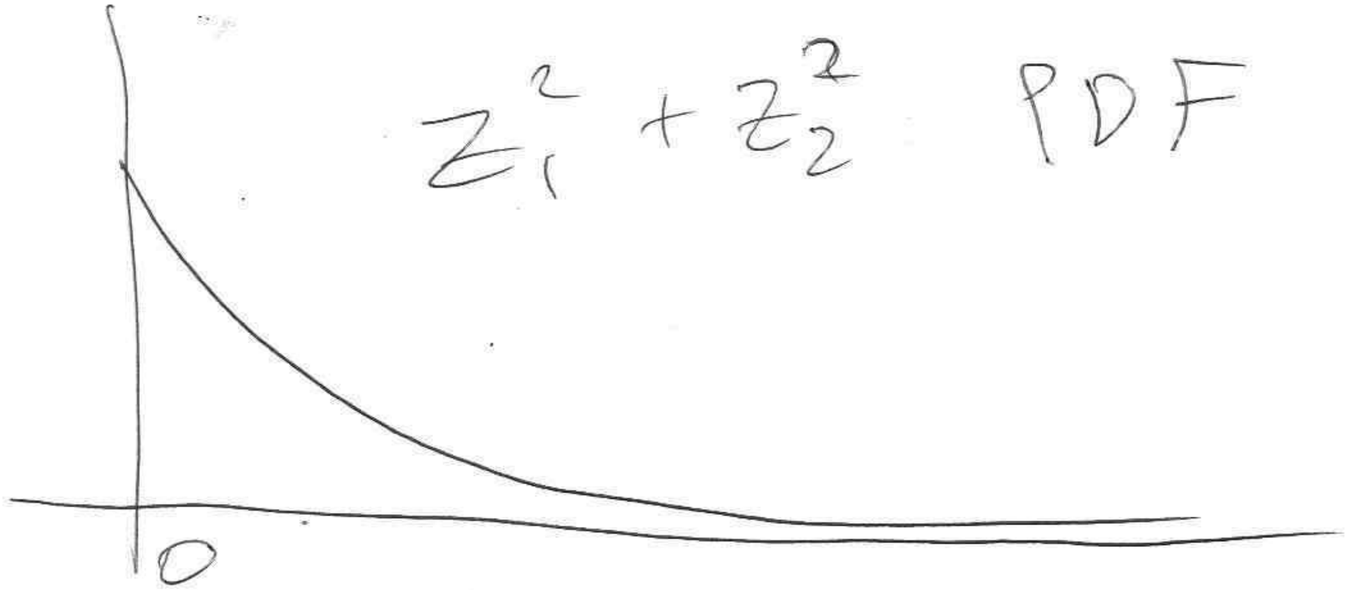




Joint PDF

$$\frac{e^{-(x^2+y^2)/2}}{2\pi} = f(x,y) \underset{\uparrow}{=} f(x)f(y)$$

independence


$Z_1^2 + Z_2^2$ PDF
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$Z_1^2 + Z_2^2 + Z_3^2$ PDF