Ch. 7 Summary: $\langle v, T^*w \rangle = \langle Tv, w \rangle$ defines T^*.

self-adjoint $(T^* = T)$ $\Longrightarrow$ **normal** $(TT^* = T^*T)$ $\Longleftarrow$ **isometry** $\|Tv\| = \|v\|$

normal $\overset{\mathbb{F}=\mathbb{C}}{\Longleftrightarrow}$ $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ for some orth. $e_1, \ldots, e_n$ as input & output basis

self-adjoint $\overset{\mathbb{F}=\mathbb{R}}{\Longleftrightarrow}$ $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ for some orth. $e_1, \ldots, e_n$ as input & output basis

isometry $\Longleftrightarrow$ $[T^* = T^{-1}]$ $\overset{\mathbb{F}=\mathbb{C}}{\Longleftrightarrow}$ $\mathcal{M}(T) = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}$ with $|z_1| = \cdots = |z_n| = 1$ for some orth basis $e_1, \ldots, e_n$ for input & output

positive $(T^* = T \;\&\; \langle Tv, v \rangle \geq 0)$ $\Longrightarrow$ self-adjoint

positive $\Longleftrightarrow$ $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ & $\lambda_1, \ldots, \lambda_n \geq 0$ For some orth. basis $e_1, \ldots, e_n$ For input & output

positive $\Longleftrightarrow$ [T has positive square root $\mathcal{M}(\sqrt{T}) = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$ For some orth basis $e_1, \ldots, e_n$ for input & output]

These all assume $T \in \mathcal{L}(V)$ and $\dim V = n < \infty$ and an inner product.

$(ST)^* = T^* S^*$

$T^{**} = T$

$(T^*T)^* = T^*T$

properties of the adjoint

$\mathcal{M}(T^*) = \overline{\mathcal{M}(T)^t}$

when using an orthonormal basis $e_1, \ldots, e_n$ for both input & output

T^*T is always positive.

Polar Decomposition:

$\forall T \in \mathcal{L}(V)\ \exists S \in \mathcal{L}(V)$

S is an isometry and

$T = S\sqrt{T^*T}$.

Singular Value Decomposition:

$\forall T \in \mathcal{L}(V)\ \exists$(orthonormal bases $e_1, \ldots, e_n$ & $f_1, \ldots, f_n$ of V)

$\forall j \leq n \quad Te_j = s_j f_j$ where

$s_j = \sqrt{\lambda_j}$ & $T^*Te_j = \lambda_j e_j$.

This all assumes $\dim V < \infty$ and an inner product on V.