Let $T \in \mathcal{L}(V,W)$ & $\dim(W) < \infty$. Then:

① T is injective $\Longleftrightarrow$ T has a left inverse $L \in \mathcal{L}(W,V)$.

② T is surjective $\Longleftrightarrow$ T has a right inverse $R \in \mathcal{L}(W,V)$

Proof. ① $\Leftarrow$: $Tx = Ty \Rightarrow x = LTx = LTy = y$.

① $\Rightarrow$: Let $w_1, \ldots, w_m$ be a basis of range(T) and extend it to a basis $w_1, \ldots, w_m, \ldots, w_n$ of W. Let $w_i = Tv_i$ for $i \leq m$. Define $L \in \mathcal{L}(W,V)$ by $Lw_i = v_i$ for $i \leq m$ and $Lw_i = 0$ for $i > m$. Given $v \in V$, we claim $LTv = v$: Let $Tv = c_1 w_1 + \cdots + c_m w_m$. By linearity, $c_1 w_1 + \cdots + c_m w_m = T(c_1 v_1 + \cdots + c_m v_m)$ and $L(cw_1 + \cdots + c_m w_m) = c_1 v_1 + \cdots + c_m v_m$. Since T is injective, $v = c_1 v_1 + \cdots + c_m v_m$. Therefore, $LTv = LT(c_1 v_1 + \cdots) = c_1 v_1 + \cdots = v$.

(2) $\Leftarrow$: For each $y \in W$, $y = T(Ry)$.

(2) $\Rightarrow$: Let $w_1, \ldots, w_n$ be a basis for W. Let $w_i = Tv_i$ for each i, which can be done because T is surjective. Define $R \in \mathcal{L}(W, V)$ by $Rw_i = v_i$. Then, by linearity, for each $c_1 w_1 + \cdots + c_n w_n \in W$ we have

$$TR(c_1 w_1 + \cdots) = c_1 TRw_1 + \cdots = c_1 w_1 + \cdots. \qquad \blacksquare$$

Given sets A, B and $T: A \to B$ with left inverse $L: B \to A$ and right inverse $R: B \to A$, we must have $L = R$.

Proof. Like in the proof of 3.54, if $b \in B$, then $L(b) = L(T(R(b))) = R(b)$ because $L(T(x)) = x$ for all $x \in A$ and $T(R(y)) = y$ for all $y \in B$.

Thus, $L = R$. $\blacksquare$