SOME BASIC ANALYSIS THEOREMS FROM A SEQUENTIAL VIEWPOINT

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For this document, we fix an ambient metric space, the complex plane $\mathbb{C}$ with its standard metric $d(z, w) = |z - w|$. All points are assumed to in $\mathbb{C}$; all sets of points are assumed to be subsets of $\mathbb{C}$.

1. Sequences and convergence.

Definition 1.1.

(1) A **subsequence** of a sequence $(z_n)_{n=0}^{\infty}$ is a sequence of the form $(z_{n_k})_{k=0}^{\infty}$ where $n_0 < n_1 < n_2 < \cdots$.
(2) A **tail** of a sequence $(z)_{n=0}^{\infty}$ is a subsequence of the form $(z_{N+n})_{n=0}^{\infty}$ (which may also be denoted by $(z_n)_{n=N}^{\infty}$).
(3) A sequence $(z_n)_{n=0}^{\infty}$ is ε-**stable** if $d(z_m, z_n) < \varepsilon$ for all $m, n \geq 0$.
(4) A sequence $(z_n)_{n=0}^{\infty}$ is ε-**close** to a point L if $d(z_n, L) < \varepsilon$ for all $n \geq 0$.
(5) A sequence $(z_n)_{n=0}^{\infty}$ is **Cauchy** if, for every $\varepsilon > 0$, it has an ε-stable tail.
(6) A sequence $(z_n)_{n=0}^{\infty}$ **converges** to L, which we often abbreviate as $z_n \to L$, if for every $\varepsilon > 0$, $(z_n)_{n=0}^{\infty}$ has a tail that is ε-close to L.

Theorem 1.2. *A sequence is Cauchy if and only if it converges to some point.*

Proof. If $z_n \to L$, then, for every $\varepsilon > 0$, $(z_n)_{n=0}^{\infty}$ has a tail that is $\frac{\varepsilon}{2}$-close to L. Such a tail is ε-stable because $d(z_m, z_n) \leq d(z_m, L) + d(L, z_n)$ for all m, n. Thus, convergence implies the Cauchy property.

To prove the converse, suppose that $(z_n)_{n=0}^{\infty}$ is Cauchy. Suppose we can prove that $\mathrm{Re}(z_n) \to A$ and $\mathrm{Im}(z_n) \to Bi$ for some reals A and B. Then, for every $\varepsilon > 0$, $(\mathrm{Re}(z_n))_{n=0}^{\infty}$ and $(\mathrm{Im}(z_n))_{n=0}^{\infty}$ will have tails $(\mathrm{Re}(z_k))_{k=M}^{\infty}$ and $(\mathrm{Im}(z_k))_{k=N}^{\infty}$ that are $\frac{\varepsilon}{2}$-close to A and B, respectively. Since $|z - w| \leq |\mathrm{Re}(z - w)| + |\mathrm{Im}(z - w)| = |\mathrm{Re}(z) - \mathrm{Re}(w)| + |\mathrm{Im}(z) - \mathrm{Im}(w)|$ for all points z and w, it follows that, taking P to be larger of M and N, the tail $(z_k)_{k=P}^{\infty}$ is ε-close to $A + Bi$.

Therefore, setting $x_n = \mathrm{Re}(z_n)$ and $y_n = \mathrm{Im}(z_n)$, it is sufficient to prove that the sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ converge to reals. Since $|\mathrm{Re}(z) - \mathrm{Re}(w)| = |\mathrm{Re}(z - w)| \leq |z - w|$ for all points z and w the sequence $(x_n)_{n=0}^{\infty}$ is Cauchy. Likwise, $(y_n)_{n=0}^{\infty}$ is Cauchy. By symmetry, it is enough to prove that the Cauchy sequence of reals $(x_n)_{n=0}^{\infty}$ converges to a real.

For each of $k = 0, 1, 2, 3, \ldots$, let $x_{n,k}$ be x_n rounded down to the nearest integer multiple of 10^{-k}. If $(x_{n,k})_{n=0}^{\infty}$ is eventually constant, then let u_k denote its final value. For each $k \geq 1$ for which u_k is defined, let d_k be the kth digit after the decimal place of u_k. First, suppose we are in the case where u_k is defined for all k. Set $u = u_0.d_1d_2d_3\ldots$. Given $\varepsilon > 0$, choose $k > 0$ such that $10^{-k} < \varepsilon$. Choose N large enough that we have $x_{n,j} = u_j$ for all $n \geq N$ and $j = 0, 1, \ldots, k$. Therefore, $u_0.d_1d_2\ldots d_k \leq x_n < u_0.d_1d_2\ldots d_k + 10^{-k}$ for all $n \geq N$, and

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$u_0.d_1d_2\dots d_k \le u \le u_0.d_1d_2\dots d_k + 10^{-k}$. Hence, $|x_n - u| \le 10^{-k} < \varepsilon$ for all $n \ge N$. Thus, for every $\varepsilon > 0$, $(x_n)_{n=0}^{\infty}$ is ε-close to u on a tail. Thus, $x_n \to u$.

Now suppose we are in the other case, that u_k is not defined for all k. Let k be least such that u_k is undefined. Suppose we are in the subcase where $k > 0$. Then

$$u_0.d_1d_2d_2\dots d_{k-1} \le x_n < u_0.d_1d_2d_2\dots d_{k-1} + 10^{k-1}$$

is eventually always true, but there are at least two values a in $\{0,\dots 9\}$ such that

$$u_0.d_1d_2d_2\dots d_{k-1}a \le x_n < u_0.d_1d_2d_2\dots d_{k-1}a + 10^{-k}$$

for infinitely many n. Let $v = u_0.d_1d_2d_2\dots d_{k-1}c$ where c is the larger of two values for a. There are then infinitely many n such that $x_n < v$ and infinitely many n such that $v \le x_n$. Given $\varepsilon > 0$, choose $m > 0$ such that $m \ge k$ and $10^{-m} \le \varepsilon$. Choose M large enough that we have $x_{n,j} = u_j$ for all $n \ge M$ and $j = 0, 1, \dots, k-1$, and such that $(x_p)_{p=M}^{\infty}$ is 10^{-m}-stable. Choose $r, s \ge M$ such that $x_r < v \le x_s$. For any $n \ge M$, we have

$$x_n = x_r + x_n - x_r \le x_r + |x_n - x_r| < x_r + 10^{-m} < v + 10^{-m} < v + \varepsilon;$$
$$x_n = x_s + x_n - x_s \ge x_s - |x_n - x_s| > x_s - 10^{-m} \ge v - 10^{-m} \ge v - \varepsilon.$$

Thus, $(x_p)_{p=M}^{\infty}$ is ε-close to v. Thus, for any $\varepsilon > 0$, $(x_n)_{n=0}^{\infty}$ has a tail ε-close to v. Thus, $x_n \to v$.

The last subcase, where $k = 0$, is handled like the previous subcase. There are at least two adjacent integer values a such that $x_n < a$ for infinitely many n and $a \le x_n < a + 1$ for infinitely many n. Letting q denote the greater of two such values, we have $x_n < q$ for infinitely many n and $q \le x_n$ for infinitely many n. Given $\varepsilon > 0$, choose $m \ge 0$ such that $10^{-m} \le \varepsilon$ and M large enough that $(x_p)_{p=M}^{\infty}$ is 10^{-m}-stable. Arguing just as in the previous paragraph, we find that $(x_p)_{p=M}^{\infty}$ is ε-close to q. Thus, $x_n \to q$. □

Theorem 1.3 (Limit Laws)**.**

(1) *If $a_n \to L$ and $b_n \to M$, then $a_n + b_n \to L + M$ and $a_nb_n \to LM$.*
(2) *If $a_n \to L$ and $b_n \to M \ne 0$, then some tail of $(a_n/b_n)_{n=0}^{\infty}$ converges to L/M.*

2. Sets of points and convergence.

Definition 2.1.

(1) A set (of points) S is **closed** if, for every convergent sequence $z_n \to L$, if every z_n is in S, then L is in S too.
(2) A set S is **complete** if, for every Cauchy sequence $(z_n)_{n=0}^{\infty}$, if every z_n is in S, then $(z_n)_{n=0}^{\infty}$ converges to some L in S.
(3) A set is **compact** if, for every sequence $(z_n)_{n=0}^{\infty}$, if every z_n is in S, then $(z_n)_{n=0}^{\infty}$ has a subsequence that converges to some L in S.
(4) A set is **bounded** if there exists some R such that $d(p, q) \le R$ for all p, q in S.
(5) A set S is **open** if, for every sequence convergent sequence $z_n \to L$, if L is in S, then a tail of $(z_n)_{n=0}^{\infty}$ consists only of points in S.

Lemma 2.2.

(1) *If $z_n \to P$ and $z_n \to Q$, then $P = Q$.*
(2) *If $z_n \to P$ and $(z_{n_k})_{k=0}^{\infty}$ is a subsequence of $(z_n)_{n=0}^{\infty}$, then $z_{n_k} \to P$.*
(3) *If $(z_{n_k})_{k=0}^{\infty}$ is a subsequence of $(z_n)_{n=0}^{\infty}$, then $n_k \ge k$ for all k.*

Theorem 2.3.

(1) *A set is complete if and only if it is closed.*
(2) *A set is compact if and only if it is closed and bounded.*
(3) *A set is closed if and only if its complement is open.*

Proof.

(1) If S is closed and $(z_n)_{n=0}^{\infty}$ is a Cauchy sequence of points in S, then $z_n \to L$ for some L; by definition of "closed," that L is in S. Thus, if S is closed, then every Cauchy sequence in S converges to a point in S. In other words, if S is closed, then it is complete. Conversely, if S is complete, z_n is in S for all n, and $z_n \to P$, then, since $(z_n)_{n=0}^{\infty}$ is convergent, it is also Cauchy; since S is complete, $z_n \to Q$ for some Q in S; since also $z_n \to P$, we have $P = Q \in S$. Thus, if S is complete, then every convergent sequence of points in S converges to a point in S. Thus, if S is complete, then S is closed.

(2) Suppose S is compact. If S were not bounded, then there would be a sequence $(z_n)_{n=0}^{\infty}$ of points in S such that $|z_n| \geq n$ for all n. But such a sequence has no 1-stable tail $(z_k)_{k=N}^{\infty}$ because, given N, $|z_{N+k} - Z_N| \geq |z_{N+k}| - |z_N| \geq N + k - |z_N| \geq 1$ if k is chosen to be sufficiently large. Therefore, a sequence $(z_n)_{n=0}^{\infty}$ as above has no Cauchy, and, hence, no convergent, subsequence. Since S is compact, there can be no such sequence. Hence S is bounded.

Moreover, S is closed because if $(w_n)_{n=0}^{\infty}$ is a sequence of points in S and $w_n \to P$, then P is in S because, letting $(v_n)_{n=0}^{\infty}$ be one of the subsequences of $(w_n)_{n=0}^{\infty}$ that converges to some Q in S, then we have $w_n \to P = Q$.

Conversely, suppose S is closed and bounded. To show that S is compact, we need to show that, given a z_n in S for all n, there exist L in S and $n_0 < n_1 < n_2 < n_3 < \cdots$ such that $z_{n_k} \to L$. Assume $(z_n)_{n=0}^{\infty}$ is as above. Since S is closed and, therefore, complete, it suffices to find $n_0 < n_1 < n_2 < n_3 < \cdots$ such that $(z_{n_k})_{k=0}^{\infty}$ is Cauchy. For future notational uniformity, let $n_{0,k} = k$ for all k. Since S is bounded, S is contained in a closed (solid) square T_0 of some finite diameter D. Divide this square into four congruent closed subsquares, each necessarily with diamater $D/2$. Since every $z_{n_{0,k}}$ is in it at least one of the four subsquares and four finite sets cannot have infinite union, we may choose T_1 from among the four subsquares such that infinitely many k are such that $z_{n_{0,k}} \in T_1$. Hence, $\left(z_{n_{0,k}}\right)_{k=0}^{\infty}$ has a subsequence $\left(z_{n_{1,k}}\right)_{k=0}^{\infty}$ consisting only of points in T_1. More generally, given $m \geq 0$, a closed square T_m with diameter $D/2^m$, and a sequence $\left(z_{n_{m,k}}\right)_{k=0}^{\infty}$ of points in T_m, we may choose a closed subsquare T_{m+1} of diamater $D/2^{m+1}$ and a subsequence $\left(z_{n_{m+1,k}}\right)_{k=0}^{\infty}$ consisting only of points from T_{m+1}. Finally, we define a "diagonal" sequence $(z_{n_m})_{m=0}^{\infty}$ by $n_m = n_{m,m}$. This sequence is a subsequence of $(z_n)_{n=0}^{\infty}$ because, for all m, $n_{m+1} > n_m$ because $n_{m+1,m+1} > n_{m+1,m} \geq n_{m,m}$ because $\left(z_{n_{m+1,k}}\right)_{k=0}^{\infty}$ is a subsequence of $\left(z_{n_{m,k}}\right)_{k=0}^{\infty}$. Moreover, for each m, $(z_{n_k})_{k=m}^{\infty}$ is $(D/2^m)$-stable because it is a subsequence of $\left(z_{n_{m,k}}\right)_{k=0}^{\infty}$ Hence, for every ε, there is an m large enough that $(z_{n_k})_{k=m}^{\infty}$ is an ε-stable tail of $\left(z_{n_j}\right)_{j=0}^{\infty}$. Thus, $(z_n)_{n=0}^{\infty}$ indeed has a Cauchy subsequence.

(3) Suppose S is closed. To show that its complement is open, we need to show that, given a point w not in S and $z_n \to w$, $(z_n)_{n=0}^{\infty}$ has a tail of points not in S. Given $(z_n)_{n=0}^{\infty}$ and w as above, if there were no such tail, then z_n would be in S for infinitely many n, in which case, $(z_n)_{n=0}^{\infty}$ would have a subsequence $(w_k)_{k=0}^{\infty}$ of points in S;

such a subsequence would necessarily converge to w. However, because S is closed, no sequence of points in S can converge a point not in S, such as w. Therefore, there is no such subsequence $(w_k)_{k=0}^{\infty}$. Therefore, $(z_n)_{n=0}^{\infty}$ must have a tail of points not in S, as desired.

Conversely, suppose S is the complement of an open set U. To show that S is closed, we need to show that, given $z_n \to w$ and z_n in S for all n, we have w in S. Given $(z_n)_{n=0}^{\infty}$ and w as above, if w were not in S, then w would be in U, so a tail of $(z_n)_{n=0}^{\infty}$ would be in U, which is impossible because every z_n is in S.

□

Theorem 2.4.

(1) *Every union of open sets is open.*
(2) *Every intersection of closed sets is closed.*
(3) *Every closed subset of a compact set is compact.*
(4) *Every intersection of finitely many open sets is open.*
(5) *Every union of finitely many closed sets is closed.*
(6) *Every union of finitely many bounded sets is bounded.*
(7) *Every union of finitely many compact sets is compact.*
(8) *If $K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ and every K_n is compact and nonempty, then $\bigcap_{n=1}^{\infty} K_n$ is nonemtpy.*

Proof.

(1) Suppose $z_n \to w \in U = \bigcup_{j\in J} U_j$ and each U_j is open. It is enough to show that $(z_n)_{n=0}^{\infty}$ has a tail in U. Then $w \in U_k$ for some $k \in J$, so a tail of $(z_n)_{n=0}^{\infty}$ is in U_k, so that tail is in U. Thus, U is open.
(2) Suppose $z_n \to w$, $z_n \in C = \bigcap_{j\in J} C_j$ for all n, and each C_j is closed. It is sufficient to show that $w \in C$. For each $j \in J$, we have $z_n \in C_j$ for all n, so $w \in C_j$. Since w is in every C_j, w is in C as desired.
(3) If $C \subseteq K$, C is closed, and K is compact, then K is bounded; hence, C is also bounded; hence, C is compact.
(4) Suppose $z_n \to w \in U = U_0 \cap U_1 \cap \cdots \cap U_{m-1}$ and each U_k is open. It is enough to show that $(z_n)_{n=0}^{\infty}$ has a tail in U. For each $k < m$, since $w \in U_k$, there exists N_k such that $z_n \in U_k$ for all $n \geq N_k$; hence, taking $N = \max\{N_k : k < m\}$, we have $z_n \in U_0 \cap U_1 \cap \cdots \cap U_{m-1}$ for all $n \geq N$. Thus, U is open.
(5) Suppose $z_n \to w$, $z_n \in C = C_0 \cup C_1 \cup \cdots \cup C_{m-1}$ for all n, and each C_j is closed. It is sufficient to show that $w \in C$. Since m finite sets cannot have infinite union, there must be some $j < m$ such that $z_n \in C_j$ for infinitely many n. Hence, $(z_n)_{n=0}^{\infty}$ has a subsequence $(w_n)_{n=0}^{\infty}$ of points in C_j. Since $w_n \to w$ and C_j is closed, $w \in C_j$. Hence, $w \in C$. Thus, C is closed.
(6) Suppose $B = B_0 \cup B_1 \cup \cdots \cup B_{n-1}$ and each B_k is bounded. For each k, choose M_k such that $d(p, q) \leq M_k$ for all $p, q \in B_k$. Let $M = \max\{M_k : k < n\}$. For each k, choose $z_k \in B_k$. Set $N = \max\{d(z_a, z_b) : a, b < n\}$. Then, for $p, q \in B$, we have, for some $a, b < n$, $p \in B_a$ and $q \in B_b$, which implies $d(p, q) \leq d(p, z_a) + d(z_a, z_b) + d(z_b, q) \leq M + N + M$. Thus, B is bounded.
(7) This immediately follows from (5) and (6).
(8) Let $z_n \in K_n$ for each n. By compactness of K_0, for some point p and some subsequence $(z_{n_k})_{k=0}^{\infty}$ of $(z_n)_{n=0}^{\infty}$, we have $z_{n_k} \to p$. For each $m = 0, 1, 2, 3, \ldots$, the tail

$(z_{n_k})_{k=m}^{\infty}$ also converges to p and lies entirely in K_m because for all $k \geq m$ we have $z_{n_k} \in K_{n_k} \subseteq K_k \subseteq K_m$ because $n_k \geq k \geq m$. Since K_m is closed, $p \in K_m$. Thus, $p \in \bigcap_{m=0}^{\infty} K_m$.

□

3. Functions, continuity, and sequences of functions.

In this section, "a function on S" is our abbreviation for "a complex-valued function f whose domain contains S."

Definition 3.1.

(1) A function is **continuous** on a set S if, for every convergent sequence $z_n \to L$, if L is in S and every z_n is in S, then $f(z_n) \to f(L)$.
(2) A function is **uniformly continuous** on a set S if, for every pair of sequences $(z_n)_{n=0}^{\infty}$, $(w_n)_{n=0}^{\infty}$ of points in S, if $d(z_n, w_n) \to 0$, then $d(f(z_n), f(w_n)) \to 0$.
(3) A sequence of functions $(f_n)_{n=0}^{\infty}$ converges **pointwise** to a function f on a set S if, for every z in S, $f_n(z) \to f(z)$.
(4) A sequence of functions $(f_n)_{n=0}^{\infty}$ converges **uniformly** to a function f on a set S if, for every sequence $(z_n)_{n=0}^{\infty}$ of points in S, $d(f_n(z_n), f(z_n)) \to 0$.

Lemma 3.2 (Subsubsequence Lemma). *Given a sequence $(z_n)_{n=0}^{\infty}$ and a point L, if every subsequence $(z_{n_k})_{k=0}^{\infty}$ has a subsequence $\left(z_{n_{k_m}}\right)_{m=0}^{\infty}$ that converges to L, then $z_n \to L$.*

Proof. We will prove the contrapositive, namely that if $z_n \not\to L$, then there is a subsequence $(z_{n_k})_{k=0}^{\infty}$ such that all of its subsequences also $\left(z_{n_{k_m}}\right)_{m=0}^{\infty}$ fail to converge to L. Assume $z_n \not\to L$. Then, for some $\varepsilon > 0$, there is no tail of $(z_n)_{n=0}^{\infty}$ that is ε-close to L. Choose such an ε. If only finitely many n were such that $d(z_n, L) \geq \varepsilon$, then taking N to be greater than all such n, we would have a tail $(z_k)_{k=N}^{\infty}$ that is ε-close to L. Since there is no such tail, there must be an infinitely many n as above. Therefore, we may list them as a sequence $n_0 < n_1 < n_2 < \cdots$ such that $d(z_{n_k}, L) \geq \varepsilon$ for all k. Thus, there is subsequence $(z_{n_k})_{k=0}^{\infty}$ of $(z_n)_{n=0}^{\infty}$ such that $d(z_{n_k}, L) \geq \varepsilon$ for all k. For any subsequence $\left(z_{n_{k_m}}\right)_{m=0}^{\infty}$ of $(z_{n_k})_{k=0}^{\infty}$, we have $d(z_{n_{k_m}}, L) \geq \varepsilon$ for all m, which implies $z_{n_{k_m}} \not\to L$. □

Lemma 3.3.

(1) *If $z_n \to L$ and $d(z_n, w_n) \to 0$, then $w_n \to L$.*
(2) *If $z_n \to L$ and $w_n \to L$, then $d(z_n, w_n) \to 0$.*
(3) *If $d(a_n, b_n) \to 0$ and $d(b_n, c_n) \to 0$, then $d(a_n, c_n) \to 0$.*

Theorem 3.4.

(1) *If S is compact and f is continuous on S, then f is uniformly continuous on S.*
(2) *If $(f_n)_{n=0}^{\infty}$ converges uniformly to f on S and every f_n is continuous on S, then f is continuous on S.*

Proof.

(1) Let S be compact and let f be continuous on S. To show uniformly continuity, we need to show that, given sequences $(z_n)_{n=0}^{\infty}$, $(w_n)_{n=0}^{\infty}$ of points in S such that $d(z_n, w_n) \to 0$, we have $d(f(z_n), f(w_n)) \to 0$. Assume $(z_n)_{n=0}^{\infty}$ and $(w_n)_{n=0}^{\infty}$ are as above. By the Subsubsequence Lemma, it suffices to show that, given $n_0 < n_1 < n_2 < \cdots$, there exist $k_0 < k_1 < k_2 < \cdots$ such that $d(z_{n_{k_m}}, w_{n_{k_m}}) \to 0$. Let

$n_0 < n_1 < n_2 < \cdots$. By compactness, we may choose L in S and a subsequence $\left(z_{n_{k_m}}\right)_{m=0}^{\infty}$ of $(z_{n_k})_{k=0}^{\infty}$ such that $z_{n_{k_m}} \to L$. Since $d(z_n, w_n) \to 0$, $d(z_{n_{k_m}}, w_{n_{k_m}}) \to 0$ too. Hence, $w_{n_{k_m}} \to L$. By continuity, $f(z_{n_{k_m}}) \to f(L)$ and $f(w_{n_{k_m}}) \to f(L)$. Hence, $d(z_{n_{k_m}}, w_{n_{k_m}}) \to 0$ as desired.

(2) Assume $(f_n)_{n=0}^{\infty}$ converges uniformly to f on S and every f_n is continuous on S. To show that f is also continuous on S, we need to show that, given a sequence $(z_n)_{n=0}^{\infty}$ of points in S that converges to a point L in S, $f(z_n) \to f(L)$. Let $(z_n)_{n=0}^{\infty}$ be as above and $n_0 < n_1 < n_2 < \cdots$. By the Subsubsequence Lemma, it is enough to find $k_0 < k_1 < k_2 < \cdots$ such that $f(z_{n_{k_m}}) \to f(L)$. For each $m = 0, 1, 2, \ldots$, f_m is continuous, so $f_m(z_{n_k}) \to f_m(L)$, so some tail $(f_m(z_{n_k}))_{k=N_m}^{\infty}$ is 2^{-m}-close to $f_m(L)$. Recursively define a sequence $k_0 < k_1 < k_2 < \cdots$ by $k_0 = N_0$ and $k_{m+1} = \max\{k_m + 1, N_{m+1}\}$. Hence, $0 \le d(f_m(L), f_m(z_{n_{k_m}})) < 2^{-m}$ for all m. Hence, $d(f_m(L), f_m(z_{n_{k_m}})) \to 0$. By uniform convergence, $d(f_m(z_{n_{k_m}}), f(z_{n_{k_m}})) \to 0$ also. Hence, $d(f_m(L), f(z_{n_{k_m}})) \to 0$. Again by uniform convergence, $d(f_m(L), f(L)) \to 0$. Hence, $d(f(L), f(z_{n_{k_m}})) \to 0$; hence, $f(z_{n_{k_m}}) \to f(L)$.

□

Theorem 3.5. *If S is compact and f is continuous on S, then $f(S)$ is compact.*

Proof. Let $z_n \in f(S)$ for all n. We need to show that $(z_n)_{n=0}^{\infty}$ has a subsequence converging to some point in $f(S)$. For each n, choose $w_n \in S$ such that $f(w_n) = z_n$. By compactness of S, there is a point $w \in S$ and a subsequence $w_{n_k} \to w$. By continuity, $z_{n_k} \to f(w)$ and $f(w) \in f(S)$, as desired. □

Theorem 3.6.

(1) *If f is continuous on S and g is continuous on $f(S)$, then $g \circ f$ is continuous on S.*
(2) *If f and g are continuous on S, then $f + g$ and fg are continuous on S.*
(3) *If f and g are continuous on S and $g \neq 0$ on S, then f/g is continuous on S.*

Proof.

(1) If $z_n \to w \in S$ and $z_n \in S$ for all n, then $f(z_n) \to f(w) \in f(S)$ and $f(z_n) \in f(S)$ for all n, which in turn implies that $g(f(z_n)) \to g(f(w))$.
(2) This is an immediate consequence of the Limit Laws.
(3) This is an immediate consequence of the Limit Laws.

□