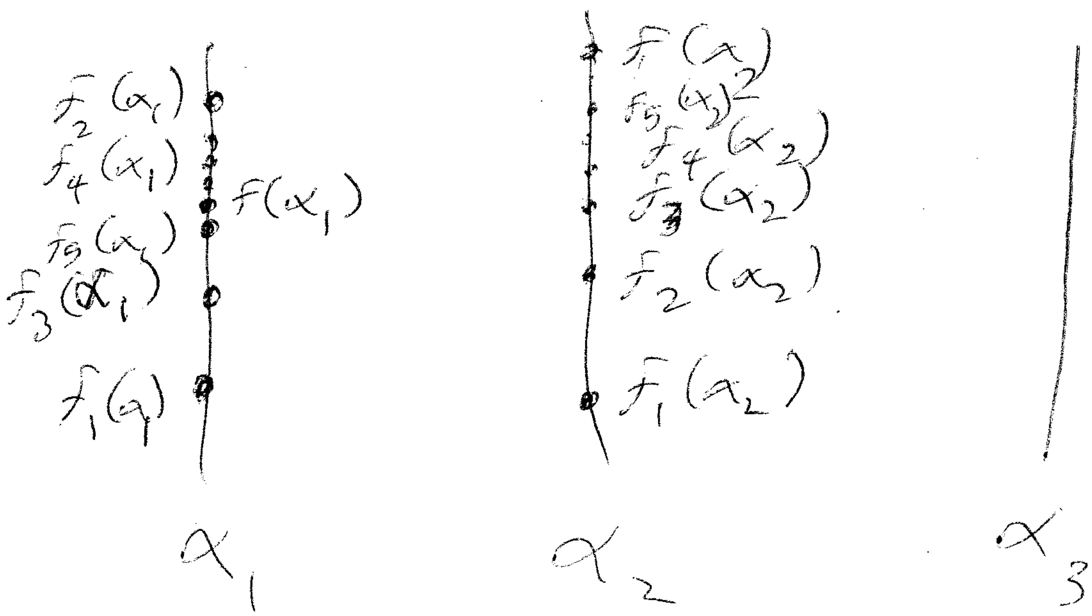
$f_1, f_2, f_3, \ldots \to f$ in a product topology. $X = \prod_{\alpha \in J} X_\alpha$

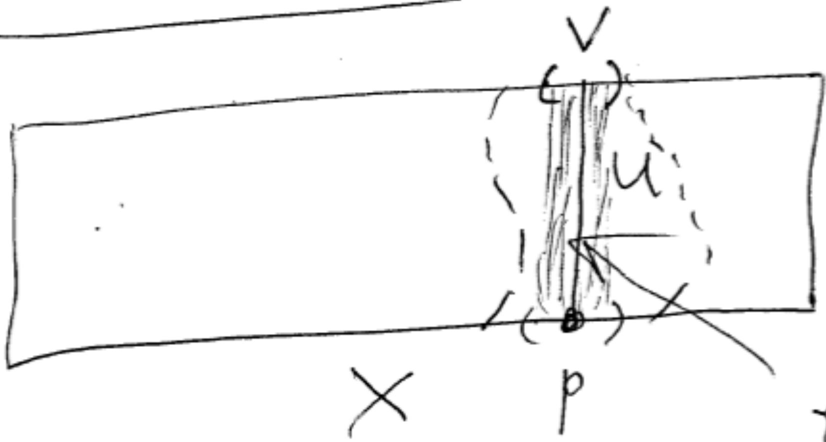
iff $\forall \alpha \in J \quad f_1(\alpha), f_2(\alpha), f_3(\alpha), \ldots$

$\to f(\alpha)$



Convergence of sequences in a product top. is <u>pointwise</u>. (43.3)

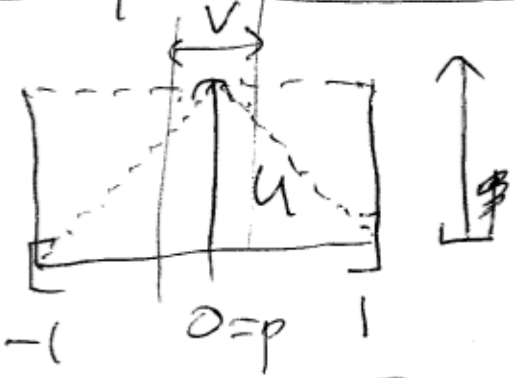
Tube Lemma (§26.8) ← Used to prove preservation of compactness by finite products without the Axiom of Choice. (§26)



Y cpct

$\{p\} \times Y \subset U$ open $\subset X \times Y$

Given p, U, X, Y as above, there is an open $V \subset X$ with $p \in V$ & $V \times Y \subset U$.



$Y = [0,1)$ $\quad \{0\} \times [0,1) \subset U$

$X = [-1,1]$

$U = \{(x,y) \in X \times Y \mid y < 1 - |x|\}$

If $0 \in V$ open $\subset [-1,1]$,

then $V \times [0,1) \not\subset U$.

So, compactness must be used in the proof of the Tube Lemma.

Property	cts image	closed subspace	products
compact	yes	yes	yes
conn.	yes	no $\{0,1\} \subset \mathbb{R}$	yes
path conn.	yes	no	yes
Seq cpct.	yes	yes	yes for ctbl prod's
	← Preservation?		
T_1	INDISCRETE TOPOLOGY	YES	
T_2			yes
T_3			yes
$T_{3.5}$			yes
T_4			yes
T_5			no
T_6			yes for ctbl prod's

Path. conn. preservation by products:

use diagonal product of paths.

Path. conn. preservation by cts. images:

$$\begin{tikzcd} {[0,1]} \arrow[r, "p", "\text{cts}"'] \arrow[rr, bend right, "f\circ p"', "\text{cts}"] & X \arrow[r, "f", "\text{cts}"'] & f(X) \end{tikzcd}$$

Compose with paths.

Preservation of seq. cpct. by closed subspaces:

$x_n \in C = \overline{C} \subset X$

$\Rightarrow x_{n_k} \to L \in \overline{C} = C$

Preservation of seq. cpct. by cts images: proof ideas.

$$\begin{tikzcd} X \arrow[r, "f", "\text{cts}"'] & f(X) \arrow[dl, "\text{id}"] \\ f(X) \arrow[u, "g"] & \end{tikzcd}$$

Let $x_n \in f(X)$

& $x_n = f(g(x_n))$

& $g(x_{k_n}) \to L$

Then $x_{k_n} \to f(L)$.