HW15

(1) [Grad only] Prove that if a linear order $(L, <)$ has the least upper bound property, then each closed interval $[a,b] = \{c \in L \mid a \leq c \leq b\}$ where $a, b \in L$ is sequentially compact. (You may* assume Munkres' Theorem 27.1 that all such $[a,b]$ are compact.) Hint: Find $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$ such that $a_1, a_2, a_3, \ldots$ or $b_1, b_2, b_3, \ldots$ is a subsequence of a given sequence $x_1, x_2, x_3, \ldots \in [a,b]$.

(2) Prove that if a linear order is nonempty but has no maximum, then its order topology is not compact.

Comment: (1) & (2) are two key steps in proving that S_Ω is sequentially compact but not compact. The third key step is this fact: if $x_1, x_2, x_3, \ldots \in S_\Omega$, then $\exists y \in S_\Omega$ $\forall n \in \mathbb{N}$ $\min(S_\Omega) \leq x_n \leq y$.

(3) Prove that $\mathbb{R}^\omega$ is not locally compact.

*I don't think this assumption will actually help your proof.

4. Prove that if X is sequentially compact and $f: X \to \mathbb{R}$ is continuous, then $f(X)$ is bounded.
Hint: Suppose $|f(x_n)| > n$ for each n.

5. Prove that if X is a compact metric space, $f: X \to X$, and $d(f(a), f(b)) = d(a,b)$ for all $a, b \in X$, then $f(X) = X$. Hint: suppose $p \in X - f(X)$ and consider p, $f(p)$, $f(f(p))$, $f(f(f(p)))$, ...