Zorn's Lemma and the Alexander Subbase Lemma

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April 24, 2018

Lemma 1 (Alexander Subbase Lemma)**.** *Let $\mathcal{S}$ be a subbase of space X such that every subcover of $\mathcal{S}$ has a finite subcover. Then X is compact.*

Proof using Zorn's Lemma. Seeking a contradiction, suppose $\mathcal{S}$ is as above but X has an open cover without a finite subcover. Let $\mathbb{O}$ denote the set of all open covers of X that lack finite subcovers. Then $\mathbb{O}$ is nonempty; hence, the union $\varnothing$ of the empty chain $\varnothing$ is a subset of some $\mathcal{U} \in \mathbb{O}$. By the following claim, each union of a nonempty chain $\mathbb{C} \subset \mathbb{O}$ is also a subset of some $\mathcal{U} \in \mathbb{O}$.

Claim. If $\mathbb{C} \subset \mathbb{O}$ is a nonempty chain, then $\bigcup \mathbb{C} \in \mathbb{O}$.

Proof. Let $\mathcal{V} \in \mathbb{C}$ and $\mathcal{W} = \bigcup \mathbb{C}$. Then $\mathcal{W}$ is a set of open subsets of X because every $\mathcal{C} \in \mathbb{C}$ is. Moreover, $\mathcal{W}$ covers X because $\mathcal{V}$ does and $\mathcal{W}$ contains $\mathcal{V}$. If $\{W_1, \ldots, W_n\}$ is a finite subcover of $\mathcal{W}$, then $W_i \in \mathcal{C}_i \in \mathbb{C}$ for some $\mathcal{C}_i$, for each i. But in this case, since $\mathbb{C}$ is a chain, $\mathcal{C}_1, \ldots, \mathcal{C}_n \subset \mathcal{C}_j$ for some j and, therefore, $\{W_1, \ldots, W_n\}$ is a finite subcover of $\mathcal{C}_j$, which is impossible because $\mathcal{C}_j \in \mathbb{C} \subset \mathbb{O}$. Therefore, $\mathcal{W}$ has no finite subcover. Thus, $\mathcal{W} \in \mathbb{O}$. $\square$

By Zorn's Lemma, $\mathbb{O}$ has a maximal element $\mathcal{M}$. Let $\mathcal{A} = \mathcal{M} \cap \mathcal{S}$. Since $\mathcal{A} \subset \mathcal{M} \in \mathbb{O}$, no finite subset of $\mathcal{A}$ covers X. But all subcovers of $\mathcal{S}$ have finite subcovers; hence, $\mathcal{A}$ does not cover X; choose $p \in X - \bigcup \mathcal{A}$. Since $\mathcal{M}$ does cover X, we may choose M such that $p \in M \in \mathcal{M}$. Since $\mathcal{S}$ is a subbase for X, we may choose $S_1, \ldots, S_n \in \mathcal{S}$ such that $p \in \bigcap_{i \le n} S_i \subset M$.

For each S_i, we have $S_i \notin \mathcal{M}$ because $S_i \in \mathcal{S} - \mathcal{A}$ because $p \in S_i - \bigcup \mathcal{A}$. Therefore, $\mathcal{B}_i = \mathcal{M} \cup \{S_i\} \notin \mathbb{O}$ by maximality of $\mathcal{M}$. But since $\mathcal{B}_i \supset \mathcal{M}$ and every $S \in \mathcal{S}$ is open in X, the set $\mathcal{B}_i$ is still an open cover of X despite $\mathcal{B}_i \notin \mathbb{O}$. Therefore, $\mathcal{B}_i$ has a finite subcover $\mathcal{F}_i$. Let $\mathcal{G}_i = (\mathcal{F}_i - \{S_i\}) \cup \{M\}$, which is a finite subset of $\mathcal{M}$ whose union contains $Y_i = (X - S_i) \cup M$.

Let $\mathcal{G} = \bigcup_{i \le n} \mathcal{G}_i$. Then $\mathcal{G}$ is a finite subset of $\mathcal{M}$ whose union contains $\bigcup_{i \le n} Y_i$, which, as demonstrated below, is all of X because $\bigcap_{i \le n} S_i \subset M$.

$$\bigcup_{i \le n} ((X - S_i) \cup M) = \left(X - \bigcap_{i \le n} S_i \right) \cup M \supset (X - M) \cup M = X$$

Thus, $\mathcal{M}$ has a finite subcover despite $\mathcal{M} \in \mathbb{O}$. Contradiction! $\square$

Lemma 2 (Zorn's Lemma)**.** *Suppose that $\mathcal{D}$ is set of sets such that, for each chain $\mathcal{C} \subset \mathcal{D}$, there exists $B \in \mathcal{D}$ such that $\bigcup \mathcal{C} \subset B$. Then $\mathcal{D}$ has a maximal element.*

Proof using the Well-Ordering Principle. Let $\leq$ be a well-ordering of $\mathcal{D}$. Define $f \colon \mathcal{D} \to \{0, 1\}$ recursively by $f(B) = 1$ if $A \subset B$ for all $A < B$ satisfying $f(A) = 1$, and $f(B) = 0$ otherwise. (If f were not well-defined, there would be a least B at which $f(B)$ was not well-defined. But then $f(A)$ would be well-defined for all $A < B$, implying that our above recursive definition of $f(B)$ actually does define $f(B)$. Therefore, f must be well-defined.)

For each pair $A, B \in \mathcal{D}$, we have $A \leq B$ or $A \geq B$. And if $A < B$ and $f(A) = f(B) = 1$, then $A \subset B$. Therefore, $f^{-1}(\{1\})$ is a chain. Therefore, we may choose $M \in \mathcal{D}$ such that $\bigcup \left(f^{-1}(\{1\})\right) \subset M$. Let us show that M is maximal in $\mathcal{D}$. Given $M \subset C \in \mathcal{D}$, it is enough to show show that $C \subset M$. Since $\bigcup \left(f^{-1}(\{1\})\right) \subset M \subset C$, we have $f(C) = 1$ by definition of f. Therefore, $C \subset \bigcup \left(f^{-1}(\{1\})\right) \subset M$. □

Remark. In the context of Z, Zermelo's axioms of set theory excluding Choice, the Axiom of the Choice, the Well-Ordering Principle, and Zorn's Lemma are all provably equivalent, but not intuitively equivalent. The old joke is that the Axiom of Choice is obviously true, the Well-Ordering Principle is obviously false, and as for Zorn's Lemma, who can say?