A chain characterization of compactness

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We begin by defining compactness using the standard closed-set characterization, which is equivalent to the usual definition in terms of open covers.[1]

Definition 1.

- A set $\mathcal{A}$ of sets has the *finite intersection property* (FIP) if $\bigcap \mathcal{F} \neq \varnothing$ for all nonempty finite $\mathcal{F} \subset \mathcal{A}$.
- A topological space X is *compact* if $\bigcap \mathcal{A} \neq \varnothing$ for every nonempty set $\mathcal{A}$ of closed subsets of X with the FIP.

Next we define "chain compactness," which more obviously matches the informal notion that a compact space is a space with no holes.

Definition 2.

- A set $\mathcal{E}$ of sets is a *chain* if, for each A and B in $\mathcal{E}$, we have $A \subset B$ or $B \subset A$.
- A topological space X is *chain compact* if $\bigcap \mathcal{C} \neq \varnothing$ for every nonempty chain $\mathcal{C}$ of nonempty closed subsets of X.

It is not hard (see proof below) to see that every chain has the FIP. So, compact implies chain compact. But the converse, that chain compact implies compact, might look doubtful because there are many examples of sets of with the FIP that are not chains, like $\{[a, a+2] \mid 0 < a < 2\}$, a set of intervals each of length 2 with common element 2. But on second thought, $0 < a < 2$ if and only if $a = 1 \pm b$ for some $0 \leq b < 1$; hence,

$$\bigcap_{0<a<2} [a, a+2] = \bigcap_{0 \leq b < 1} \Big([1-b, (1-b)+2] \cap [1+b, (1+b)+2] \Big) = \bigcap_{0 \leq b < 1} [1+b, 3-b]$$

and $\{[1+b, 3-b] \mid 0 \leq b < 1\}$ is a chain. Maybe behind every set with the FIP hides a chain in way that allows us to deduce compactness from mere chain compactness? Assuming the Axiom of Choice, the proof below demonstrates exactly this.

[1]See, for example, Theorem 26.9 in Munkres' textbook for a proof of this equivalence.

Theorem 3. *A space X is compact if and only if it is chain compact.*

While the proof below might be new, I strongly doubt the above theorem has never been proved before. For experts in order theory, the above theorem is an easy corollary of an old theorem[2] that a poset with a minimum element is chain-complete if and only if it is directed-complete. In contrast, the proof below is relatively elementary, using a well-ordering but not not requiring knowledge of ordinals, posets, or Zorn's Lemma.

Proof. First, given X compact, we will prove that X is chain compact. Suppose $\mathcal{C}$ is a nonempty chain of nonempty closed subsets of X. We will show that $\bigcap \mathcal{C} \neq \varnothing$. We start by showing that $\mathcal{C}$ has the FIP. Suppose that $C_1, \dots, C_n \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, for some permutation σ of $\{1, \dots, n\}$, $C_{\sigma(1)} \supset C_{\sigma(2)} \supset \cdots \supset C_{\sigma(n)}$. Therefore, $\bigcap_{i=1}^{n} C_i = C_{\sigma(n)}$, which is nonempty because $C_{\sigma(n)} \in \mathcal{C}$. Thus, $\mathcal{C}$ has the FIP. Hence, since X is compact, $\bigcap \mathcal{C} \neq \varnothing$. Thus, X is chain compact.

Second, given Y chain compact, we will prove that Y is compact. Suppose that $\mathcal{A}$ is a nonempty set of closed subsets of Y with the FIP. We will show that $\bigcap \mathcal{A} \neq \varnothing$. By Zermelo's Theorem, $\mathcal{A}$ has a well-ordering. (We assume the Axiom of Choice.) Let $<$ be such an ordering. Since $\mathcal{A}$ is nonempty, it has a minimum element A_0 with respect to our well-ordering. If $\mathcal{A}$ does not also have a maximum, then replace $<$ with $<'$ where $A <' A_0$ for all instances of $A_0 < A$, and $A <' B$ for all instances of $A_0 < A < B$. The new $<$ will still be a well-ordering and will have a maximum. For each $A \in \mathcal{A}$, define $I_1(A) = \bigcap_{B \leq A} B$. This makes $I_1(\max(\mathcal{A})) = \bigcap \mathcal{A}$; hence, it is enough to prove that $I_1(\max(\mathcal{A}) \neq \varnothing$. Seeking a contradiction, assume $I_1(\max(\mathcal{A})) = \varnothing$.

It is not hard to see that $\{I_1(A) \mid A < \max(\mathcal{A})\}$ is a chain. And $I_1(\max(\mathcal{A})) = \varnothing$ makes this chain a bad chain, by which I mean that the chain looks like it might be a counterexample to Y's chain compactness. However, to find an actual counterexample, we will need to work harder, essentially shrinking our bad chain to a minimal bad chain.

Suppose that $n \in \mathbb{N}$ and we have defined $I_n(A)$ for all $A \in \mathcal{A}$ such that $I_n(\max(\mathcal{A})) = \varnothing$ just as $I_1(\max(\mathcal{A})) = \varnothing$. Define $I_{n+1}(A)$ for each $A \in \mathcal{A}$ as follows. The set $\mathcal{B}_n = \{B \in \mathcal{A} \mid I_n(B) = \varnothing\}$ is nonempty because it contains $\max(\mathcal{A})$. Because $<$ is a well-ordering, $\mathcal{B}_n$ has a minimum which we will call M_n. Define $I_{n+1}(A) = I_n(A) \cap M_n$ for all $A \in \mathcal{A}$. This definition makes

$$I_{n+1}(\max(\mathcal{A})) = \varnothing \cap M_n = \varnothing$$

just as $I_n(\max(\mathcal{A})) = \varnothing$. Therefore, by induction, our above definition scheme, applied to every $n \in \mathbb{N}$, produces a well-defined sequence $(I_n(A))_{n \in \mathbb{N}}$ for each $A \in \mathcal{A}$.

Claim. There exists $n \in \mathbb{N}$ such that $M_n = M_{n+1}$.

Proof. Because $<$ is a well-ordering, $\{M_n \mid n \in \mathbb{N}\}$ has a minimum, say, M_s. In particular, $M_s \leq M_{s+1}$. We will show that $M_{s+1} \leq M_s$ also. We have

[2]This theorem is in, for example, P.M. Cohn's 1965 textbook *Universal Algebra.*

$I_s(M_s) = \varnothing$ because $M_s \in \mathcal{B}_s$. We also have $M_s \in \mathcal{B}_{s+1}$ because

$$I_{s+1}(M_s) = I_s(M_s) \cap M_s = \varnothing \cap M_s = \varnothing.$$

Therefore, $M_{s+1} = \min(\mathcal{B}_{s+1}) \leq M_s$. $\square$

Choose k such that $M_k = M_{k+1}$. Let $\mathcal{C} = \{I_{k+1}(A) \mid A < M_{k+1}\}$. Since Y is chain compact, to reach a contradiction it is enough to prove that $\mathcal{C}$ is a nonempty chain of nonempty closed sets such that $\bigcap \mathcal{C} = \varnothing$. First, by definition of M_{k+1}, each $C \in \mathcal{C}$ is nonempty.

Second, let us that prove that $\mathcal{C} \neq \varnothing$. For each $A \in \mathcal{A}$, since $I_1(A) = \bigcap_{B \leq A} B$ and $I_{n+1}(A) = I_n(A) \cap M_n$ for each n, we have

$$I_{j+1}(A) = \left(\bigcap_{B \leq A} B \right) \cap M_1 \cap M_2 \cap M_3 \cap \cdots \cap M_j \tag{1}$$

for all $j \in \mathbb{N}$. The set $\mathcal{C}$ is empty only if there is no $A < M_{k+1}$. But if M_{k+1} is indeed the minimum of $\mathcal{A}$, then (1) implies

$$I_{k+1}(M_{k+1}) = M_{k+1} \cap M_1 \cap \cdots \cap M_k. \tag{2}$$

The left side of (2) is empty because $M_{k+1} \in \mathcal{B}_{k+1}$. The right side of (2) is nonempty because $\mathcal{A}$ has the FIP. Thus, (2) is false. So, $\mathcal{C} \neq \varnothing$.

Third, each $C \in \mathcal{C}$ is closed because (1) implies that each $I_{k+1}(A)$ is the intersection of closed sets, namely, $M_1, \ldots, M_k$ and all $B \leq A$, which are closed because they are in $\mathcal{A}$.

Fourth, let us show that $\mathcal{C}$ is a chain. For this it is enough to show that $I_{k+1}(E) \subset I_{k+1}(F)$ whenever $F < E$. Suppose $F < E$. If $p \in B$ for all $B \leq E$, then $p \in B$ for all $B \leq F$. Therefore, $I_1(E) \subset I_1(F)$. Intersecting both sides of $I_1(E) \subset I_1(F)$ with $M_1 \cap \cdots \cap M_k$, we obtain $I_{k+1}(E) \subset I_{k+1}(F)$.

Fifth and finally, we prove $\bigcap \mathcal{C} = \varnothing$. Using (1) and $M_k = M_{k+1}$, we obtain

$$\begin{aligned}
\bigcap \mathcal{C} &= \bigcap_{A < M_{k+1}} I_{k+1}(A) \\
&= \bigcap_{A < M_k} \left[\left(\bigcap_{B \leq A} B \right) \cap \bigcap_{i=1}^{k} M_i \right] \\
&= \left(\bigcap_{B < M_k} B \right) \cap \bigcap_{i=1}^{k} M_i \\
&= \left(\bigcap_{B \leq M_k} B \right) \cap \bigcap_{i=1}^{k-1} M_i \\
&= I_k(M_k) \\
&= \varnothing.
\end{aligned}$$

$\square$