

A sequential approach to uniform continuity and compact metric spaces

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Summary. We take some standard concepts and theorems about metric spaces that use “ ε ,” prove the concepts equivalent to a sequential concepts that replace “ ε ” and any explicit inequalities with subsequences and limits, and prove the theorems uses these subsequences.

- Continuity, uniform continuity, Cauchy sequences, totally bounded metric spaces, and compact metric spaces are characterized in terms of subsequences.
- Subsequences are used to prove that uniformly continuous images preserve the Cauchy property and total boundedness.
- Subsequences are used to prove that the compact metric spaces are exactly the complete totally bounded metric spaces.
- Subsequences are used to prove the Uniform Continuity Theorem.

Definition 1. Given a topological space X , $p \in X$, and a sequence $c_1, c_2, c_3, \dots \in X$, we write $c_n \rightarrow p$ if for every neighborhood U of p there exists $N \in \mathbb{N}$ such that $c_N, c_{N+1}, c_{N+2}, \dots \in U$.

Definition 2. Given a map between $f: X \rightarrow Y$ between metric spaces,

- f is *uniformly continuous* if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$;
- f is *sequentially uniformly continuous* if

$$d_X(a_n, b_n) \rightarrow 0 \text{ implies } d_Y(f(a_n), f(b_n)) \rightarrow 0.$$

Lemma 1. A map between $f: X \rightarrow Y$ between metric spaces is uniformly continuous if and only if it is sequentially uniformly continuous.

Proof. First, we assume uniform continuity and prove sequential uniform continuity. Given $d_X(a_n, b_n) \rightarrow 0$ and $\varepsilon > 0$, it is enough to find a tail of distances $(d_Y(f(a_n), f(b_n)))_{n \geq N}$ all $< \varepsilon$. Choose $\delta > 0$ such that, for all $a, b \in X$,

$d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$. Since $d_X(a_n, b_n) \rightarrow 0$, there is a tail of distances $(d_X(a_n, b_n))_{n \geq N}$ all $< \delta$. Therefore $d_Y(f(a_n), f(b_n)) < \varepsilon$ for all $n \geq N$.

Now we assume sequential uniform continuity and prove uniform continuity. Given $\varepsilon > 0$, it is enough to find $\delta > 0$ such that $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \varepsilon$. Seeking a contradiction, suppose there is no such δ . For each $n \in \mathbb{N}$, case $\delta = \frac{1}{n}$ allows us to choose $a_n, b_n \in X$ such that $d_X(a_n, b_n) < \frac{1}{n}$ but $d_Y(f(a_n), f(b_n)) \geq \varepsilon$. Therefore, $d_X(a_n, b_n) \rightarrow 0$ but $d_Y(f(a_n), f(b_n)) \not\rightarrow 0$, in contradiction with sequential uniform continuity. \square

Definition 3. We say that sequence b_1, b_2, b_3, \dots is a *subsequence* of sequence a_1, a_2, a_3, \dots if there exist $n_1 < n_2 < n_3 < \dots$ such that $a_{n_k} = b_k$ for all $k \in \mathbb{N}$.

Definition 4. Suppose a_1, a_2, a_3, \dots is a sequence in a metric space.

- Given $\varepsilon > 0$, we say a_1, a_2, a_3, \dots is ε -*stable* if $d(a_m, a_n) < \varepsilon$ for all $m, n \in \mathbb{N}$.
- We say a_1, a_2, a_3, \dots is *Cauchy* if, for every $\varepsilon > 0$, there is an ε -stable tail $a_N, a_{N+1}, a_{N+2}, \dots$
- We say a_1, a_2, a_3, \dots is *sequentially Cauchy* if $d(b_n, c_n) \rightarrow 0$ for all pairs of subsequences b_1, b_2, b_3, \dots and c_1, c_2, c_3, \dots

Lemma 2. A sequence a_1, a_2, a_3, \dots is Cauchy if and only if it is sequentially Cauchy.

Proof. First, assume a_1, a_2, a_3, \dots is Cauchy. Given subsequences $b_k = a_{m_k}$ and $c_k = a_{n_k}$, we will show that $d(b_k, c_k) \rightarrow 0$. To do this, we assume $\varepsilon > 0$ and then find a tail $d(b_K, c_K), d(b_{K+1}, c_{K+1}), d(b_{K+2}, c_{K+2}), \dots$ all less than ε . By assumption, there is an L such that $a_L, a_{L+1}, a_{L+2}, \dots$ is ε -stable. Choose K large enough that $m_K, n_K \geq L$. Then $d(b_k, c_k) < \varepsilon$ for all $k \geq K$. Thus, a_1, a_2, a_3, \dots is sequentially Cauchy.

Now instead assume that a_1, a_2, a_3, \dots is sequentially Cauchy. We will show it is Cauchy. Seeking a contradiction, suppose it is not, that $\varepsilon > 0$ and no tail $a_N, a_{N+1}, a_{N+2}, \dots$ is ε -stable. We will reach a contradiction by finding subsequences p_1, p_2, p_3, \dots and q_1, q_2, q_3, \dots such that $d(p_k, q_k) \geq \varepsilon$ for all $k \in \mathbb{N}$. Let $s_1 = t_1 = 1$ and then, given $n \in \mathbb{N}$ and given $s_1 < t_1 < s_2 < t_2 < \dots < s_n < t_n$, choose $t_{n+1} > s_{n+1} > t_n$ such that $d(a_{s_{n+1}}, a_{t_{n+1}}) \geq \varepsilon$, which is possible because the tail $a_{t_n+1}, a_{t_n+2}, a_{t_n+3}, \dots$ is not ε -stable. The resulting subsequences $p_k = a_{s_k}$ and $q_k = a_{t_k}$ are as desired. \square

Theorem 1. If $f: X \rightarrow Y$ is a uniformly continuous map between metric spaces and a_1, a_2, a_3, \dots is Cauchy in X , then $f(a_1), f(a_2), f(a_3), \dots$ is Cauchy in Y .

Proof. Given subsequences p_1, p_2, p_3, \dots and q_1, q_2, q_3, \dots of $f(a_1), f(a_2), f(a_3), \dots$, say, $p_k = f(a_{m_k})$ and $q_k = f(a_{n_k})$ where $m_1 < m_2 < m_3 < \dots$ and $n_1 < n_2 < n_3 < \dots$, let $b_k = a_{m_k}$ and $c_k = a_{n_k}$. Then $d(b_k, c_k) \rightarrow 0$ because a_1, a_2, a_3, \dots is Cauchy. Since f is uniformly continuous, $d(p_k, q_k) = d(f(b_k), f(c_k)) \rightarrow 0$ too. \square

Definition 5.

- A metric space X is *totally bounded* if, for each $\varepsilon > 0$, X has a finite open covering by ε -balls: $X = \bigcup_{i \leq n} B(x_i, \varepsilon)$.
- A metric space X is *sequentially totally bounded* if every sequence has a Cauchy subsequence.

Lemma 3. *A metric space X is totally bounded if and only if it is sequentially totally bounded.*

Proof. First we assume sequential total boundedness and prove total boundedness. Let $\varepsilon > 0$. Assuming that X is not covered by finitely many ε -balls, we will prove a contradiction. Define a sequence x_1, x_2, x_3, \dots by choosing $x_n \in X - \bigcup_{i < n} B(x_i, \varepsilon)$ for each $n \in \mathbb{N}$. Let $n_1 < n_2 < n_3 < \dots$ be such that $y_k = x_{n_k}$ is a Cauchy subsequence. The sequence y_1, y_2, y_3, \dots has an ε -stable tail $y_K, y_{K+1}, y_{K+2}, \dots$; hence, $y_{K+1} \in B(y_K, \varepsilon)$, which contradicts $x_{n_{K+1}} \in X - \bigcup_{i < n_{K+1}} B(x_i, \varepsilon)$.

Now we assume total boundedness and prove sequential total boundedness. Suppose $a_1, a_2, a_3, \dots \in X$. For each $n \in \mathbb{N}$, let $F_n \subset X$ be finite and such that $X = \bigcup_{x \in F_n} B(x, \frac{1}{n})$. Construct a chain $I_1 \supset I_2 \supset I_3 \supset \dots$ of infinite subsets of \mathbb{N} as follows. Let $I_0 = \mathbb{N}$. Given infinite $I_{n-1} \subset \mathbb{N}$ infinite, choose, for each $i \in I_{n-1}$, a point $f(i) \in F_n$ such that $a_{f(i)} \in B(f(i), \frac{1}{n})$. Since I_{n-1} is infinite and F_n is finite, f must send infinitely many inputs to the same output. Choose $x_n \in F_n$ and $I_n = f^{-1}(\{x_n\})$ such that I_n is infinite. Then $I_{n-1} \supset I_n$ as required.

Now choose $n_1 < n_2 < n_3 < \dots$ as follows. Given $m \in \mathbb{N}$ and n_i for all $i < m$, choose n_m from the infinite set I_m such that $n_m > n_i$ for all $i < m$. We claim that the resulting subsequence $b_m = a_{n_m}$ is Cauchy. To prove this, we will find, given $\varepsilon > 0$, an ε -stable tail $b_M, b_{M+1}, b_{M+2}, \dots$. Choose M such that $\frac{1}{M} \leq \frac{\varepsilon}{2}$. Suppose $M \leq i, j$. Then $n_i, n_j \in I_M$. Hence, $d(b_i, b_j) \leq d(b_i, x_M) + d(x_M, b_j) < \varepsilon$. Thus, $b_M, b_{M+1}, b_{M+2}, \dots$ is ε -stable. \square

Theorem 2. *If $f: X \rightarrow Y$ is uniformly continuous and X is totally bounded, then $f(X)$ is totally bounded.*

Proof. Assume $f: X \rightarrow Y$ is uniformly continuous and X is totally bounded. Let $y_1, y_2, y_3 \dots \in f(X)$ and choose x_n such that $y_n = f(x_n)$, for each $n \in \mathbb{N}$. By assumption, there are $n_1 < n_2 < n_3 < \dots$ such that w_1, w_2, w_3, \dots is Cauchy where $w_m = x_{n_m}$. By Theorem 1, setting $z_m = f(w_m)$ makes z_1, z_2, z_3, \dots a Cauchy subsequence of $y_1, y_2, y_3 \dots$. Thus, $f(X)$ is totally bounded. \square

Lemma 4. *A convergent sequence in a metric space is Cauchy.*

Proof. Given $a_n \rightarrow x$ in a metric space and subsequences b_k and c_k , we have $b_k \rightarrow x$ and $c_k \rightarrow x$, which together imply $d(b_k, c_k) \leq d(b_k, x) + d(x, c_k) \rightarrow 0$. \square

Lemma 5. *If a_1, a_2, a_3, \dots is a Cauchy sequence with convergent subsequence $b_k = a_{n_k} \rightarrow x$, then $a_n \rightarrow x$.*

Proof. Since a_1, a_2, a_3, \dots is sequentially Cauchy, $d(a_k, b_k) \rightarrow 0$. Therefore, $a_k \rightarrow x$. \square

Definition 6.

- A metric space X is *complete* if every Cauchy sequence converges.
- A topological space is *compact* if every open cover has a finite subcover.
- A metric space is *sequentially compact* if every sequence has a convergent subsequence.

Lemma 6. *A metric space X is sequentially compact if and only if it is sequentially totally bounded and complete.*

Proof. If X is sequentially totally bounded and complete, then every sequence has a Cauchy subsequence which converges, making X sequentially compact. To prove the converse, suppose that X is sequentially compact. Then every sequence has a convergent subsequence which is Cauchy, making X totally bounded. To see that X is also complete, suppose that a_1, a_2, a_3, \dots is Cauchy. By assumption, there is a convergent subsequence $b_m = a_{n_m} \rightarrow x$. By Lemma 5, a_1, a_2, a_3, \dots converges. \square

Lemma 7. *A metric space X is sequentially compact if and only if it is compact.*

Proof. First, suppose X is compact. Then X is totally bounded because if $\varepsilon > 0$ then $\{B(x, \varepsilon) \mid x \in X\}$ is an open cover with a finite subcover. Therefore, to show that X is sequentially compact, it suffices to show that X is complete. So, given a_1, a_2, a_3, \dots Cauchy, we will show that $a_n \rightarrow p$ for some $p \in X$. Let $C = \bigcap_{N \in \mathbb{N}} C_N$ where $C_N = \{a_n \mid n \geq N\}$. Since each C_N is closed and nonempty and $C_1 \supset C_2 \supset C_3 \supset \dots$, compactness of X implies that C is nonempty. Choose $p \in C$. Given $\varepsilon > 0$, choose N such that $a_N, a_{N+1}, a_{N+2}, \dots$ is $\varepsilon/2$ -stable. Since $p \in C \subset C_N$, the neighborhood $B(p, \varepsilon/2)$ of p intersects $\{a_n \mid n \geq N\}$. So, choose $M \geq N$ such that $d(p, a_M) < \varepsilon/2$. Then $d(a_n, p) \leq d(a_n, a_M) + d(a_M, p) < \varepsilon$ for all $n \geq N$. Thus, $a_n \rightarrow p$.

Now instead suppose that X is sequentially compact. Given an open cover \mathcal{U} of X without a finite subcover, we will prove a contradiction. For each $n \in \mathbb{N}$, let $F_n \subset X$ be finite and such that $X = \bigcup_{x \in F_n} B(x, \frac{1}{n})$. For each n , if $B(x, \frac{1}{n})$ were covered by a finite $\mathcal{G}_x \subset \mathcal{U}$ for each $x \in F_n$, then $\bigcup_{x \in F_n} \mathcal{G}_x$ would be a finite subcover of \mathcal{U} . So, choose $x_n \in F_n$ such that $B(x_n, \frac{1}{n})$ is not covered by any finite $\mathcal{G} \subset \mathcal{U}$. By sequential compactness, there are z and $n_1 < n_2 < n_3 < \dots$ such that $y_k \rightarrow z$ where $y_k = x_{n_k}$. Then $B(z, \delta) \subset U \in \mathcal{U}$ for some δ, U . Choosing k large enough that $\frac{1}{n_k} \leq \frac{\delta}{2}$ and $d(y_k, z) \leq \frac{\delta}{2}$, we have

$$B\left(y_k, \frac{1}{n_k}\right) \subset B(z, \delta) \subset U \in \mathcal{U},$$

which contradicts $B\left(x_{n_k}, \frac{1}{n_k}\right)$ not being covered by any finite $\mathcal{G} \subset \mathcal{U}$. \square

Theorem 3. *A metric space X is compact if and only if it is totally bounded and complete.*

Proof. Combine Lemmas 6 and 7. □

Lemma 8 (Subsubsequence Lemma). *Given a topological space X , $p \in X$, and a sequence $c_1, c_2, c_3, \dots \in X$, suppose that, for every subsequence $b_m = c_{n_m}$ where $n_1 < n_2 < n_3 < \dots$, there exists a subsubsequence $a_k = b_{m_k}$ where $m_1 < m_2 < m_3 < \dots$ such that $a_k \rightarrow p$. Then $c_n \rightarrow p$.*

Proof. Seeking a contradiction, suppose that $c_n \not\rightarrow p$. Then there is a neighborhood U of p such that U does not contain any tail $c_N, c_{N+1}, c_{N+2}, \dots$. Choose $n_1 < n_2 < n_3 < \dots$ by the following method of selecting n_m for each $m \in \mathbb{N}$. Given $m \in \mathbb{N}$ and n_i for $i < m$, choose $N > n_i$ for all $i < m$, observe that $\{c_N, c_{N+1}, c_{N+2}, \dots\} \not\subset U$, and then choose $n_m \geq N$ such that $c_{n_m} \notin U$.

By construction, the subsequence b_1, b_2, b_3, \dots where $b_m = c_{n_m}$ avoids U . By assumption, b_1, b_2, b_3, \dots has a subsequence $a_k = b_{m_k} \rightarrow p$. Hence, some tail $a_K, a_{K+1}, a_{K+2}, \dots$ is in U . But $a_K = b_{m_K} \notin U$ by construction. Contradiction! □

Definition 7. Given a function $f: X \rightarrow Y$ between topological spaces,

- f is *continuous* if $p \in \overline{A}$ implies $f(p) \in \overline{f(A)}$;
- f is *sequentially continuous* if $a_n \rightarrow p$ implies $f(a_n) \rightarrow f(p)$.

Lemma 9. *Given a function $f: X \rightarrow Y$ between metric spaces, f is continuous if and only if it is sequentially continuous.*

Proof. Suppose f is sequentially continuous. Given $p \in \overline{A} \subset X$, choose $a_n \in A \cap B(p, \frac{1}{n})$ for each $n \in \mathbb{N}$. Then $a_n \rightarrow p$. Hence, $f(a_n) \rightarrow f(p)$. Hence, $f(p) \in \overline{\{f(a_n) \mid n \in \mathbb{N}\}} \subset \overline{f(A)}$.

Now suppose f is continuous. Given $a_n \rightarrow p \in X$, we will show that $f(a_n) \rightarrow f(p)$. Let $b_m = a_{n_m}$ where $n_1 < n_2 < n_3 < \dots$. By the Subsubsequence Lemma, it is enough to find $m_1 < m_2 < m_3 < \dots$ such that $f(c_k) \rightarrow f(p)$ where $c_k = b_{m_k}$. Choose m_1, m_2, m_3, \dots as follows. Given $k \in \mathbb{N}$ and m_i for $i < k$, let $S = \{b_m \mid m_i < m \text{ for all } i < k\}$. Since every neighborhood of p contains a tail $a_N, a_{N+1}, a_{N+2}, \dots$, every neighborhood of p intersects S . Therefore, $p \in \overline{S}$. Hence, $f(p) \in \overline{f(S)}$; hence, $f(S) \cap B(f(p), \frac{1}{k}) \neq \emptyset$. Therefore, we may choose m_k such that $d(f(b_{m_k}), f(p)) < \frac{1}{k}$ and such that $m_k > m_i$ for all $i < k$. Therefore, $d(f(c_k), f(p)) < \frac{1}{k} \rightarrow 0$, which implies $f(c_k) \rightarrow f(p)$. □

Theorem 4. *If $f: X \rightarrow Y$ is a continuous map between metric spaces and X is compact, then f is uniformly continuous.*

Proof. Given $f: X \rightarrow Y$ continuous and $d_X(a_n, b_n) \rightarrow 0$, we will show that $d_Y(f(a_n), f(b_n)) \rightarrow 0$. Let $p_m = a_{n_m}$ and $q_m = b_{n_m}$ where $n_1 < n_2 < n_3 < \dots$. By the Subsubsequence Lemma, it is enough to find $m_1 < m_2 < m_3 < \dots$ such that $d_Y(f(r_k), f(s_k)) \rightarrow 0$ where $r_k = p_{m_k}$ and $s_k = q_{m_k}$. Since X

is compact, we may choose $x \in X$ and $m_1 < m_2 < m_3 < \dots$ such that $r_k \rightarrow x$. Since $d_X(a_n, b_n) \rightarrow 0$, we also have $d_X(r_k, s_k) \rightarrow 0$. Therefore, $s_k \rightarrow x$. Since f is continuous, $f(r_k) \rightarrow f(x)$ and $f(s_k) \rightarrow f(x)$. Therefore, $d_Y(f(r_k), f(s_k)) \rightarrow 0$. \square