Limits involving ∞

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1 Limits at ∞

Rational functions sometimes have horizontal asymptotes. We find and describe these asymptotes using limits at ∞ (and limits at $-\infty$). The standard technique is to divide the numerator and denominator by the largest power of x in the denominator.

The following function has a horizontal asymptote at $y = 2$.

$$\lim_{x\to\infty} \frac{x - 4x^5 + 1}{-2x^5 - 7} = \lim_{x\to\infty} \frac{(x - 4x^5 + 1/x^5)}{(-2x^5 - 7)/x^5} = \lim_{x\to\infty} \frac{\frac{1}{x^4} - 4 + \frac{1}{x^5}}{-2 - \frac{7}{x^5}} = \frac{0 - 4 + 0}{-2 - 0} = 2$$

Note that $\lim_{x\to-\infty} \frac{x-4x^5+1}{-2x^5-7} = 2$ also. To see this, you use exactly the same technique.

$$\lim_{x\to-\infty} \frac{x - 4x^5 + 1}{-2x^5 - 7} = \lim_{x\to-\infty} \frac{(x - 4x^5 + 1/x^5)}{(-2x^5 - 7)/x^5} = \lim_{x\to-\infty} \frac{\frac{1}{x^4} - 4 + \frac{1}{x^5}}{-2 - \frac{7}{x^5}} = \frac{0 - 4 + 0}{-2 - 0} = 2$$

The next function has a horizontal asymptote at $y = 0$.

$$\lim_{x\to\infty} \frac{3x + 1}{x^2 - 1} = \lim_{x\to\infty} \frac{(3x + 1)/x^2}{(x^2 - 1)/x^2} = \lim_{x\to\infty} \frac{\frac{3}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{0 + 0}{1 - 0} = 0$$

Again, we get the same limit as $x \to -\infty$.

$$\lim_{x\to-\infty} \frac{3x + 1}{x^2 - 1} = \lim_{x\to-\infty} \frac{(3x + 1)/x^2}{(x^2 - 1)/x^2} = \lim_{x\to-\infty} \frac{\frac{3}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{0 + 0}{1 - 0} = 0$$

If you don't want to have to write so much twice, then use following notation most concise.

$$\lim_{x\to\pm\infty} \frac{3x + 1}{x^2 - 1} = \lim_{x\to\pm\infty} \frac{(3x + 1)/x^2}{(x^2 - 1)/x^2} = \lim_{x\to\pm\infty} \frac{\frac{3}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{0 + 0}{1 - 0} = 0$$

Some non-rational functions have "one-sided" horizontal asymptotes. For example, $\lim_{x\to\infty} \frac{x|x|}{x^2+5} = 1$ and $\lim_{x\to-\infty} \frac{x|x|}{x^2+5} = -1$. (Exercise: why?)

Here's a rational function without any horizontal asymptote.

$$\lim_{x\to\infty} \frac{3x^5 + 5x - 8}{-6x^2 - 1} = \lim_{x\to\infty} \frac{(3x^5 + 5x - 8)/x^2}{(-6x^2 - 1)/x^2} = \lim_{x\to\infty} \frac{3x^3 + \frac{5}{x} - \frac{8}{x^2}}{-6 - \frac{1}{x^2}} = \frac{3(+\text{big})^3 + 0 - 0}{-6 - 0} = \frac{+\text{big}}{-6} = -\infty$$

Note that the limit as $x \to -\infty$ is slightly different.

$$\lim_{x\to\infty} \frac{3x^5 + 5x - 8}{-6x^2 - 1} = \lim_{x\to\infty} \frac{(3x^5 + 5x - 8)/x^2}{(-6x^2 - 1)/x^2} = \lim_{x\to\infty} \frac{3x^3 + \frac{5}{x} - \frac{8}{x^2}}{-6 - \frac{1}{x^2}} = \frac{3(-\text{big})^3 + 0 - 0}{-6 - 0} = \frac{-\text{big}}{-6} = \infty$$

2 Vertical asymptotes

Division by zero produce finite limits when the zero divisors are cancelled by factors of the numerator.

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-4x+3}=\lim_{x\to 3}\frac{(x+1)(x-3)}{(x-1)(x-3)}=\lim_{x\to 3}\frac{x+1}{x-1}=\frac{3+1}{3-1}=2$$

When cancellation fails, there is a vertical asymptote and the limit fails to exist. The interesting question is how the limit fails to exist. The general pattern is that if $f(x)\to L\neq 0$ and $g(x)\to 0$ as $x\to a$, then $f(x)/g(x)$ gets very large as x approaches a. But does $f(x)/g(x)$ get large and positive or large and negative as $x\to a$? Both? Does the answer change if x only approaches a from the left or the right?

The following function gets large and positive as x approaches 3 from the right, but gets large and negative as x approaches 3 from the left.

$$\lim_{x\to 3^+}\frac{x^2-2x-3}{x^2-6x+9}=\lim_{x\to 3^+}\frac{(x+1)(x-3)}{(x-3)^2}=\lim_{x\to 3^+}\frac{x+1}{x-3}=\frac{(3+\text{small})+1}{(3+\text{small})-3}=\frac{4}{+\text{small}}=\infty$$

$$\lim_{x\to 3^-}\frac{x^2-2x-3}{x^2-6x+9}=\lim_{x\to 3^-}\frac{(x+1)(x-3)}{(x-3)^2}=\lim_{x\to 3^-}\frac{x+1}{x-3}=\frac{(3-\text{small})+1}{(3-\text{small})-3}=\frac{4}{-\text{small}}=-\infty$$

The following function gets large and negative as x approaches 3 from both directions.

$$\lim_{x\to 3}\frac{x^2+1}{-x^2+6x-9}=\lim_{x\to 3}\frac{x^2+1}{-(x-3)^2}=\frac{(3\pm\text{small})^2+1}{-(\pm\text{small})^2}=\frac{10}{-\text{small}}=-\infty$$

An alternative way to compute the above limits is to make sign tables.

		-1		3	
$x+1$	$-$	0	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	0	$+$
$\frac{x+1}{x-3}$	$+$	0	$-$	$*$	$+$

Thus, $\lim_{x\to 3^+}\frac{x+1}{x-3}=\infty$ because for x close to 3 from the right, $\frac{x+3}{x-1}$ is postive. By similar reasoning, $\lim_{x\to 3^-}\frac{x+1}{x-3}=-\infty$.

		3	
x^2+1	$+$	$+$	$+$
$-(x-3)^2$	$-$	0	$-$
$\frac{x^2+1}{-(x-3)^2}$	$-$	$*$	$-$

From the above table we see that $\lim_{x\to 3}\frac{x^2+1}{-(x-3)^2}=-\infty$.