Yet another optimization problem

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We're bidding for a contract to manufacture natural gas tanks. The volumne of each tank will be 1000 cubic feet. As is typical, the shape of the gas tank will be a circular cylinder capped by hemispheres on both ends. In addition to a fixed cost of $800 per tank, our cost per square foot of tank surface area is $4 for the cylindrical part of the tank and $6 for the spherical part. What is our minimal cost for making a such a tank?

First, let's get the geometry out of the way. Let r denote the radius (in feet) of the circular cylinder, which is also the radius of the hemispheres. Let x denote the length (in feet) of the cylinder. The circumference of the cylinder is $2\pi r$, so the surface area of the cylinder is $2\pi rx$. The volume of the cylinder is $\pi r^2 x$ The surface area of the hemispheres add up to the surface area of a sphere of radius r, which is $4\pi r^2$. The volume of the hemispheres is the volume of a sphere with radius r, which is $\frac{4}{3}\pi r^3$.

Thus, the total volume of the tank is $\pi r^2 x + \frac{4}{3}\pi r^3$, which is constrained to equal 1000. The cost C (in dollars) of the tank is $800 + 4(2\pi rx) + 6(4\pi r^2)$. Solving our constraint for x (because solving for r is much harder), we find that

$$x = \frac{1000}{\pi r^2} - \frac{4r}{3}.$$

After plugging this formula for x into our cost formula, we find that the cost in terms of r alone is given by

$$C = 800 + \frac{8000}{r} - \frac{32\pi r^2}{3} + 24\pi r^2.$$

After simplifying, we have $C = 800 + 8000/r + 40\pi r^2/3$. To optimize C, we differentiate with respect to r: $dC/dr = -8000/r^2 + 80\pi r/3$. We next solve $dC/dr = 0$ for r to find the critical points:

$$\begin{aligned} 0 &= -\frac{8000}{r^2} + \frac{80\pi r}{3} \\ \frac{8000}{r^2} &= \frac{80\pi r}{3} \\ 24000 &= 80\pi r^3 \\ \left(\frac{300}{\pi}\right)^{1/3} &= r \end{aligned}$$

Thus, the unique critical point is $r \approx 4.57078$ feet. Plugging this value into our formula for x, we get $x \approx 9.14156$ feet. Plugging these values into our original cost formula, we get approximately $3,425.37.

Is this truly the minimal cost? If we differentiate C again, we get

$$\frac{d^2C}{dr^2} = \frac{16000}{r^3} + \frac{80\pi}{3},$$

which is always positive if r is positive. Thus, for all physically meaningful r, our cost function is concave up, so $3,425.37 is indeed a local minimum and the global minimum.