221 Calculus, Fall 2007, Section 306/308

Homework 5 (Due in class November 20)

1 Exercise Solve #67 on page 354 of Thomas' *Calculus*.

2 Exercise Let $f(x) = (\sin^5 \sqrt{x})/\sqrt{x}$ for $x > 0$ and $f(0) = 0$. Prove that f is continuous on $[0, \infty)$. Then find the average value of f on $[0, \pi^2]$. (Hint: $\sin^5 \theta = (\sin^2 \theta)^2 \sin \theta$ and $\cos^2 \theta + \sin^2 \theta = 1$.)

3 Exercise Solve #56 on page 408 of Thomas' *Calculus*.

4 Exercise Solve part (d) of #24 on page 415 of Thomas' *Calculus*.

5 Exercise Find the length of the spiral curve given by $x = (t^2 - 1) \cos t$ and $y = (t^2 - 1) \sin t$ where t ranges from 0 to 6π.

6 Exercise (Optional) The length $B_1(r)$ of the interval $[-r, r]$ is equal to the following integral.

$$\int_{-r}^{r} dx$$

Clearly, the integral is equal to $2r$.

The area $B_2(r)$ of a disc with radius r is equal to the following integral.

$$\int_{-r}^{r} B_1\left(\sqrt{r^2 - y^2}\right) dy$$

(Why? The integral is the area of the region $\left\{(x, y) : -\sqrt{r^2 - y^2} \le x \le \sqrt{r^2 - y^2}\right\}$, which is the same region as $\{(x, y) : x^2 + y^2 \le r^2\}$.) Show that the integral is equal to πr^2. (Hint: use the substitution $y = r \sin \theta$.)

The volume $B_3(r)$ of a ball with radius r is equal to the following integral.

$$\int_{-r}^{r} B_2\left(\sqrt{r^2 - z^2}\right) dz$$

(Why? If we take the ball with radius r and center $(0, 0, 0)$ and take the slice formed by intersecting it with the plane described by $z = z_0$, then we get a disc with radius $\sqrt{r^2 - z_0^2}$ (assuming $|z_0| \le r$).) Show that the integral is equal to $\frac{4}{3}\pi r^3$.

The "volume" $B_4(r)$ of a four-dimensional ball with radius r is calculated by following integral.

$$\int_{-r}^{r} B_3\left(\sqrt{r^2 - w^2}\right) dw$$

(The four-dimensional ball with radius r and center $(0, 0, 0, 0)$ is defined to be the region $\{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 \le r^2\}$.) Show that the integral is equal to $\frac{1}{2}\pi^2 r^4$.

Calculate the "volumes" of a five-dimensional ball with radius r and a six-dimensional ball with radius r.