221 Calculus, Fall 2007, Section 306/308

Rules for 0 and ∞

The following equations are true and meaningful.

$$7 + \infty = \infty \tag{1}$$
$$42 - \infty = -\infty \tag{2}$$
$$\infty + \infty = \infty \tag{3}$$
$$-\infty - \infty = -\infty \tag{4}$$
$$4 \cdot \infty = \infty \tag{5}$$
$$6 \cdot (-\infty) = -\infty \tag{6}$$
$$(-22) \cdot (-\infty) = \infty \tag{7}$$
$$\infty \cdot \infty = \infty \tag{8}$$
$$(-\infty) \cdot \infty = -\infty \tag{9}$$
$$(-\infty) \cdot (-\infty) = \infty \tag{10}$$
$$\frac{17}{\infty} = 0 \tag{11}$$
$$\frac{-8}{\infty} = 0 \tag{12}$$
$$\frac{0}{\infty} = 0 \tag{13}$$
$$\frac{2}{-\infty} = 0 \tag{14}$$
$$\frac{-3}{-\infty} = 0 \tag{15}$$
$$\frac{0}{-\infty} = 0 \tag{16}$$

And what meaning are these equations full of? They are facts about limits. For example, equation (3) means that if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then $\lim_{x\to c}(f(x) + g(x)) = \infty$.

The following expressions are not meaningful. If you encounter of these types of expressions in trying to calculate a limit, then treat it as an error message, telling you that you need to try a different approach to find the limit (if it exists).

$$\infty - \infty, \qquad 0 \cdot \infty, \qquad \frac{\infty}{\infty},$$
$$\frac{\infty}{0}, \qquad \frac{-\infty}{0},$$
$$\frac{0}{0}, \qquad \frac{6}{0}, \qquad \frac{-4}{0}$$

Intuitively, dividing one by zero should give you ∞ or $-\infty$. The problem is that for a limit to exist, it has to be one or the other, not both. For rational functions, we deal with division by zero by taking a limit from the left or from the right. However, this approach does not work in general:

$$\lim_{x\to 0^+} \frac{1}{x \sin(1/x)} \text{ does not exist. (Exercise: why?)}$$