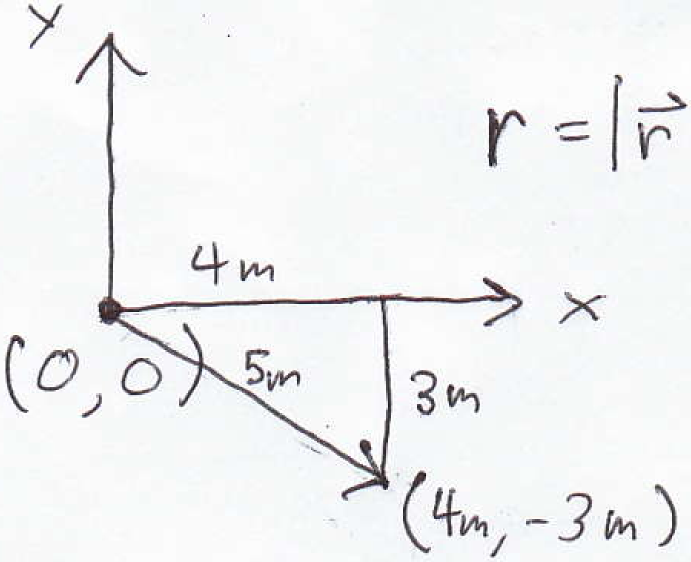
Ch. 3

Vectors & 2D/3D kinematics

vector	2D	3D	describing motion
$\vec{r}$	(r_x, r_y)	(r_x, r_y, r_z)	position/ displacement



$$r = |\vec{r}| = \begin{cases} \sqrt{r_x^2 + r_y^2} & (2D) \\ \sqrt{r_x^2 + r_y^2 + r_z^2} & (3D) \end{cases}$$

distance from (0,0) / (0,0,0)

$$\vec{r} = (4m, -3m) \Rightarrow r = 5m$$

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad v_x = \frac{dr_x}{dt} \qquad v_y = \frac{dr_y}{dt}$$

displacement from time $t=a$ to $t=b$:

$$\Delta\vec{r} = \int_{t=a}^{t=b} d\vec{r} = \int_{t=a}^{t=b} \vec{v}\, dt$$

$$\text{distance travelled} = \int_{t=a}^{t=b} |d\vec{r}| = \int_{t=a}^{t=b} |\vec{v}|\, dt$$

$$\vec{a} = \frac{d\vec{v}}{dt} \qquad a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt}$$

$$\Delta\vec{v} = \int_{t=a}^{t=b} d\vec{v} = \int_{t=a}^{t=b} \vec{a}\, dt$$

average velocity: $\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{\int_{t=a}^{t=b} \vec{v}\, dt}{b-a}$

average acceleration: $\dfrac{\Delta\vec{v}}{\Delta t} = \dfrac{\int_{t=a}^{t=b} \vec{a}\, dt}{b-a}$

Gravity near earth's surface:

$$\vec{a} = (0, \underbrace{-9.80\,m/s^2}_{g})$$

Find $\vec{v}$ at time τ given $\vec{a}$ as above & $\vec{v}$ at time 0.

$\Delta\vec{v}$ from $t=0$ to $t=\tau$ is

$$\Delta\vec{v} = \vec{v}\Big|_{t=\tau} - \vec{v}\Big|_{t=0} = \int_{t=0}^{t=\tau} \vec{a}\, dt$$

constant $(0, -9.80)\,m/s^2$

$$\Delta\vec{v} = \vec{a}\int_{t=0}^{t=\tau} dt$$

$$\Delta\vec{v} = \vec{a}\, t\Big|_{t=0}^{t=\tau} = \vec{a}(\tau - 0) = \vec{a}\,\tau$$

$$\vec{v}\Big|_{t=\tau} - \vec{v}\Big|_{t=0} = \vec{a}\,\tau$$

$$\vec{v}\Big|_{t=\tau} = \vec{v}\Big|_{t=0} + \vec{a}\,\tau$$

Like $v = v_0 + at$
from Ch. 2
for constant a.

Now find $\vec{r}$ at time τ given $\vec{a}$ & $\vec{v}\Big|_{t=0}$ as before & $\vec{r}\Big|_{t=0}$.

$$\Delta\vec{r} = \vec{r}\Big|_{t=\tau} - \vec{r}\Big|_{t=0} = \int_{t=0}^{t=\tau} \underbrace{\vec{v}\,dt}_{d\vec{r}}$$

$$\Delta\vec{r} = \int_{t=0}^{t=\tau} \underbrace{\left(\vec{v}\Big|_{t=0} + \vec{a}\,t\right)}_{\vec{v}\text{ at time } t\text{, not } \tau} dt$$

$$\Delta\vec{r} = \left[\left(\vec{v}\Big|_{t=0}\right)t + \frac{1}{2}\vec{a}\,t^2\right]_{t=0}^{t=\tau}$$

$$\Delta\vec{r} = \left[\left(\vec{v}\Big|_{t=0}\right)\tau + \frac{1}{2}\vec{a}\,\tau^2\right] - [0 + 0]$$

Also $\Delta\vec{r} = \vec{r}\Big|_{t=\tau} - \vec{r}\Big|_{t=0}$

$$\vec{r}\Big|_{t=\tau} = \vec{r}\Big|_{t=0} + \Delta\vec{r}$$

$$\vec{r}\Big|_{t=\tau} = \vec{r}\Big|_{t=0} + \left(\vec{v}\Big|_{t=0}\right)\tau + \frac{1}{2}\vec{a}\tau^2$$

Compare to $x = x_0 + v_0 t + \frac{1}{2}at^2$
from Ch. 2 for constant acceleration.

Example: $\vec{r}\Big|_{t=0} = (0,0)$

$$\vec{v}\Big|_{t=0} (15, 18)\ m/s$$

$$\vec{a} = (0, -9.8\ m/s^2) \qquad \boxed{(m/s)/s}$$

$$\vec{v}\Big|_{t=\tau} = (15,18)\,m/s + (0, -9.8\overbrace{m/s^2}^{(m/s)/s})\,\tau$$

$$\vec{v} = (15\,m/s, (18 - 9.8\,\tau/s)\,m/s)$$

E.g. at $\tau = 3.0\,s$,

$$\vec{v} = (15\,m/s, -11\,m/s)$$

$$\vec{r}\Big|_{t=\tau} = (0,0)\,\mathrm{m} + (15, 18)(\mathrm{m/s})\tau$$
$$+ \frac{1}{2}(0, -9.8\,\mathrm{m/s^2})\,\tau^2$$

$$\vec{r} = (0 + 15\tau\,\mathrm{m/s} + \tfrac{1}{2}(0)\tau^2,$$
$$0 + 18\tau\,\mathrm{m/s} + \tfrac{1}{2}(-9.8\,\mathrm{m/s^2})\tau^2)$$

$$\vec{r} = (15\tau/\mathrm{s},\ 18\tau/\mathrm{s} - 4.9\tau^2/\mathrm{s}^2)\,\mathrm{m}$$

E.g. at $\tau = 3.0\,\mathrm{s}$:

$\vec{r} = (45,$ [illegible] $4)\,\mathrm{m}$

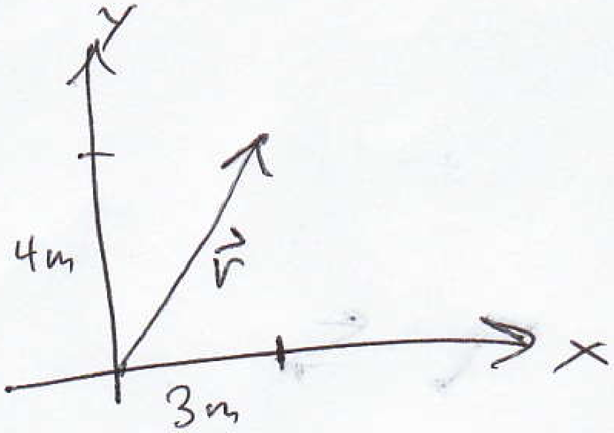
Direction & angles w/ vectors

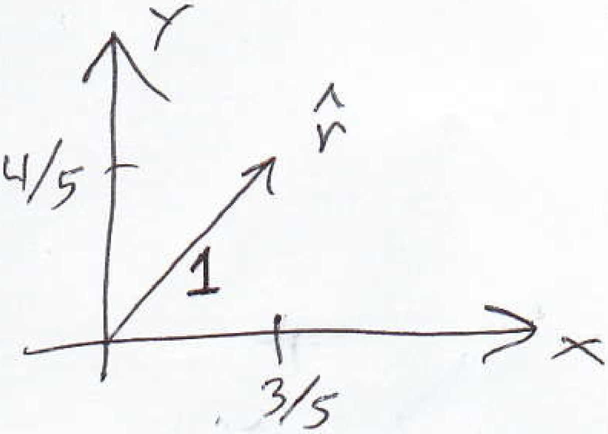
$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r} = \text{vector direction of } \vec{r}$$

E.g., $\vec{r} = (3m, 4m)$:

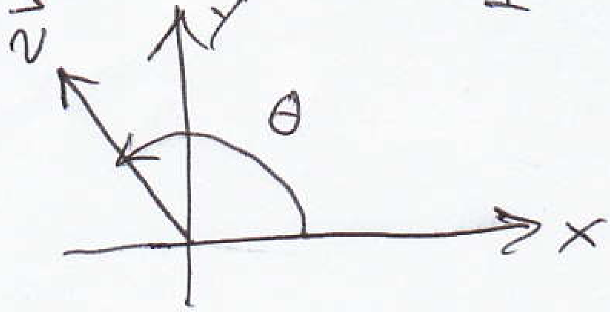
$$\hat{r} = \frac{(3m, 4m)}{5m} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$|\hat{r}| = \frac{|\vec{r}|}{|\vec{r}|} = 1 = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$





$\hat{r}$ = "$\underbrace{\text{unit}}_{1}$ vector in the direction of $\vec{r}$"



$$(\cos\theta, \sin\theta) = \hat{v}$$

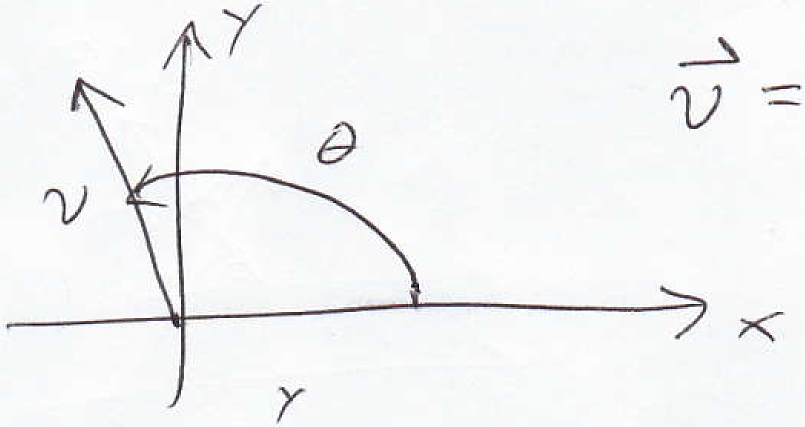
$$\vec{v} = (-2\,m/s,\ 5\,m/s)$$

$$(\cos\theta, \sin\theta) = \frac{(-2\,m/s,\ 5\,m/s)}{\sqrt{(-2m/s)^2 + (5m/s)^2}}$$

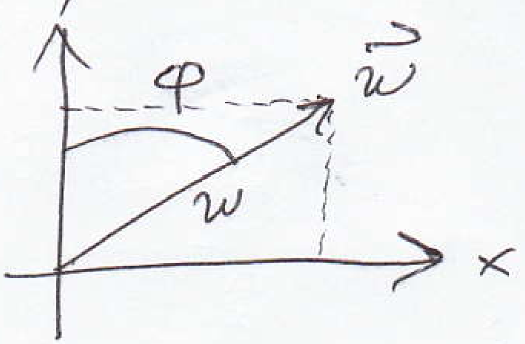
$$\cos\theta = \frac{-2}{\sqrt{29}}, \quad \sin\theta = \frac{5}{\sqrt{29}}$$

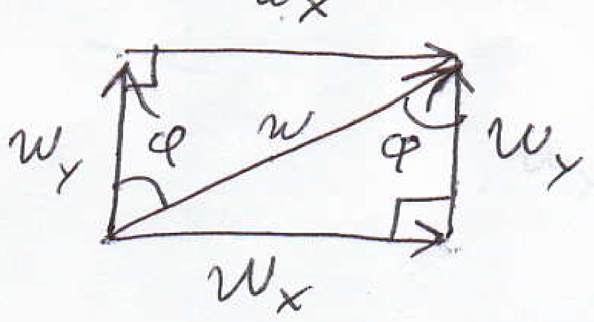
$$\cos\theta = \frac{v_x}{\sqrt{v_x^2 + v_y^2}} \qquad \sin\theta = \frac{v_y}{\sqrt{v_x^2 + v_y^2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{v_y}{v_x}$$



$$\vec{v} = (\underbrace{v\cos\theta}_{v_x}, \underbrace{v\sin\theta}_{v_y})$$





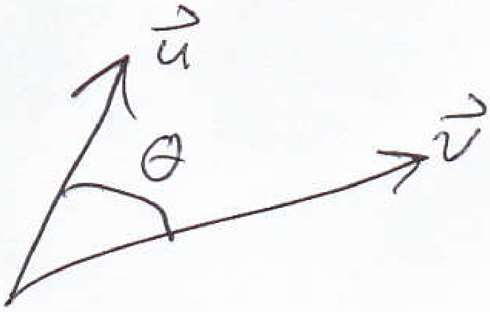
$$\sin\varphi = \frac{w_x}{w} \Rightarrow w_x = w\sin\varphi$$

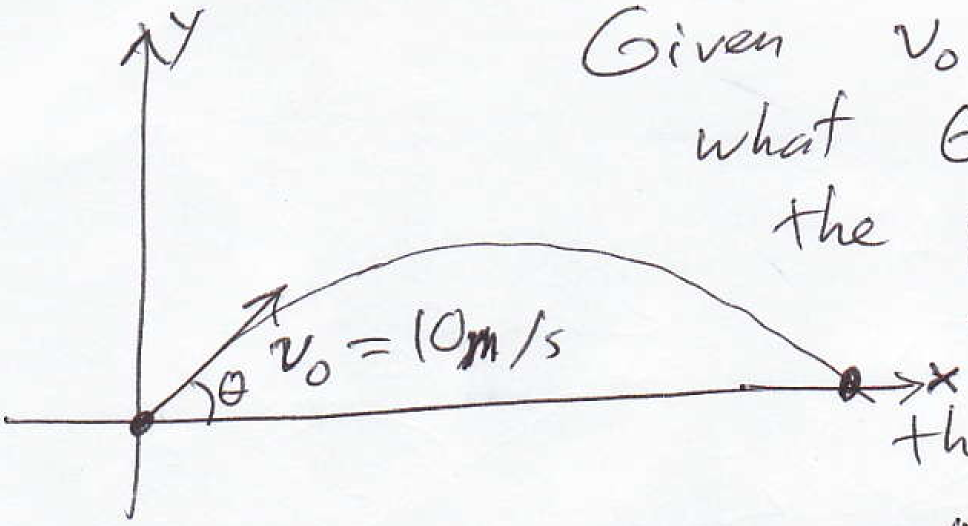
$$\cos\varphi = \frac{w_x}{w} \Rightarrow w_y = w\cos\varphi$$

Angle θ between $\vec{u}$ & $\vec{v}$:

$$\cos\theta = \frac{(u_x v_x + u_y v_y)}{\sqrt{(u_x^2+u_y^2)(v_x^2+v_x^2)}}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$





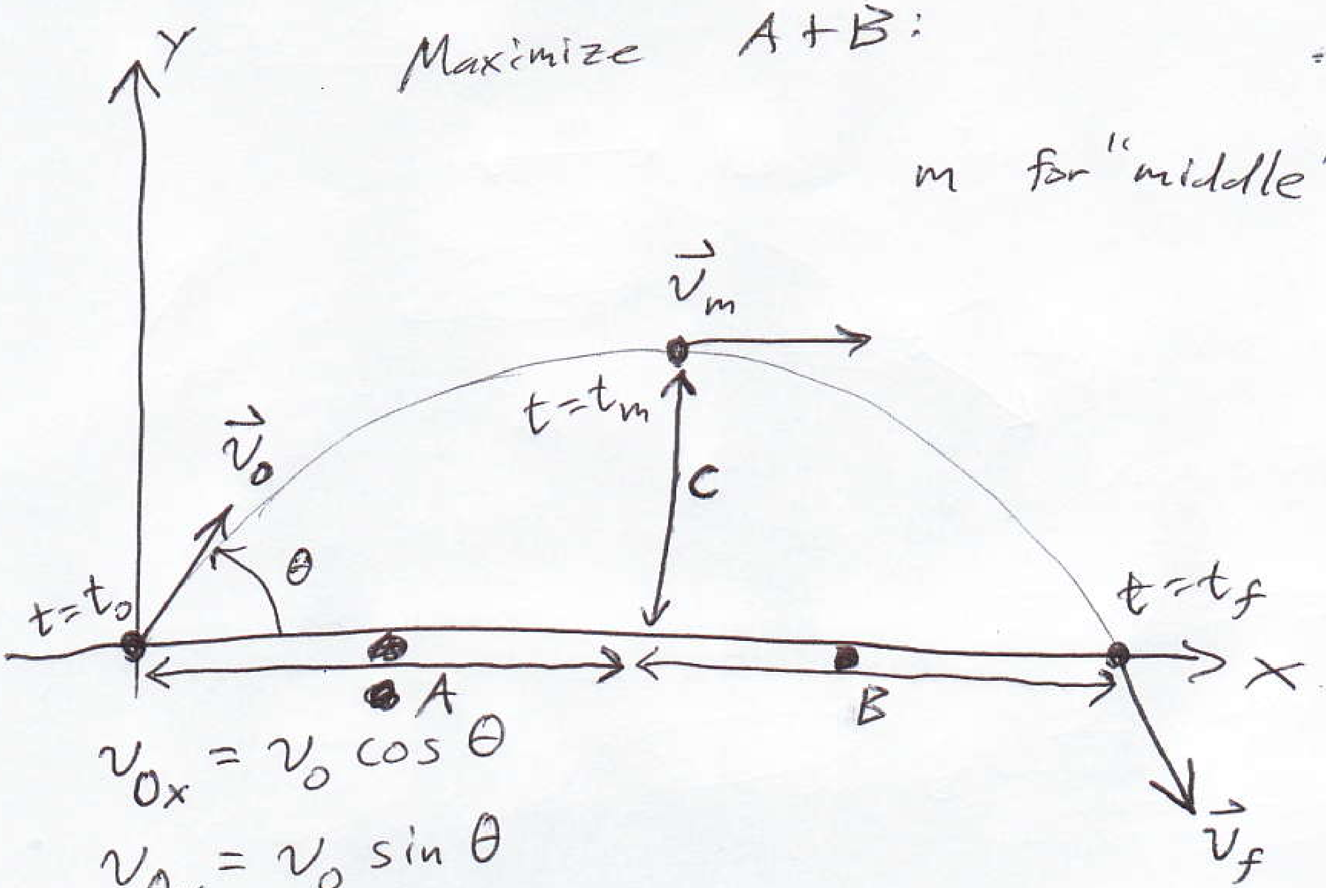
Given $v_0 = 10$ m/s, what θ sends the projectile furthest in the x-direction on impact?

Shortcuts for <u>constant</u> acceleration:

$$\Delta(v_x) = a_x \Delta t$$
$$\Delta(v_y) = a_y \Delta t$$
$$\Delta(v_x^2) = 2 a_x \Delta r_x$$
$$\Delta(v_y^2) = 2 a_y \Delta r_y$$

Maximize $A + B$:

m for "middle"



$v_{0x} = v_0 \cos\theta$

$v_{0y} = v_0 \sin\theta$

f for "final"

$v_{mx} = v_m$

$v_{my} = 0$

$v_{fx} = ?$ →

$v_{fy} = ?$

$$\boxed{\begin{aligned} &a_x = 0, \text{ so } v_x \text{ is constant,} \\ &\text{so } v_{fx} = v_{mx} = v_{0x}, \text{ so} \\ &v_{fx} = v_m = v_0 \sin\theta. \end{aligned}}$$

Since v_x is constant,

$$A = r_{mx} - r_{0x} = \int_{t_0}^{t_m} v_x \, dt = v_x t \Big|_{t=t_0}^{t=t_m}$$

$$= v_x (t_m - t_0).$$

Note: $v_x = v_{0x} = v_{mx} = v_{fx}$.

Similarly, $B = v_x (t_f - t_m)$.

$$\cancel{v_{my}^2 - v_{oy}^2 = 2a_y(r_{my} - r_{oy}) = 2a_y C}$$

$$\cancel{v_0^2 \sin \theta = \ldots = 2gC}$$

Instance of $\Delta v_y = a_y \Delta t$

$$v_{my} - v_{oy} = a_y(t_m - t_o)$$

Since $A = v_x(t_m - t_o)$, $t_m - t_o = \dfrac{A}{v_x}$, so

$\underbrace{v_{my}}_{0} - v_{oy} = a_y A / v_x$, so $A = \dfrac{-v_{oy} v_x}{a_y}$.

Similarly, $t_f - t_m = \dfrac{B}{v_x}$ and

$v_{fy} - \underbrace{v_{my}}_{0} = a_y(t_f - t_m) = a_y B / v_x$, so

$B = \dfrac{v_{fy} v_x}{a_y}$, so $A + B = \dfrac{(v_{fy} - v_{oy}) v_x}{a_y}$

We know $a_y = -g = -9.80 \text{ m/s}^2$;

we know $v_{oy} = v_0 \sin \theta$ & $v_x = v_{ox} = v_0 \cos \theta$;

$v_{fy} = ?$ Use $\Delta v_x^2 = 2a_y \Delta r_y$:

$$v_{fy}^2 - \underbrace{v_{my}^2}_{0^2} = 2a_y(r_{fy} - r_{my}) = -2a_y C$$

$$v_{fy} = \pm\sqrt{-2a_y C}$$

$$\underbrace{v_{my}^2}_{0^2} - v_{oy}^2 = 2a_y(r_{my} - r_{oy}) = 2a_y C.$$

$$v_{oy} = \pm\sqrt{-2a_y C} = \pm v_{fy},$$

so $v_{fy} = \pm v_{oy}$. We know

$v_{oy} = v_o \sin\theta$ is positive & v_{fy} is

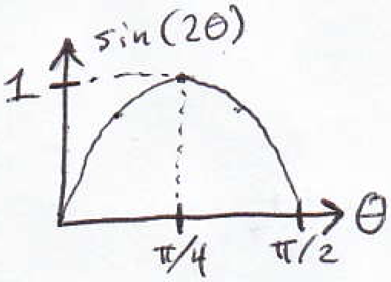
negative, so $v_{fy} = -v_{oy} = -v_o \sin\theta$.

$$A + B = \frac{(v_{fy} - v_{oy})v_x}{a_y} = \frac{(-2v_{oy})v_x}{-g}$$

$$A + B = \frac{2v_{oy}v_{ox}}{g} = \frac{2v_o^2(\sin\theta)(\cos\theta)}{g}$$

$2\sin\theta\cos\theta = \sin 2\theta$

$$A + B = [v_o^2 \sin(2\theta)]/g.$$



To maximize $A+B$, maximize $\sin(2\theta)$ with $0° < \theta < 90°$, i.e., $0 < \theta < \frac{\pi}{2}$.

$0 = (\sin(2\theta))' = 2\cos 2\theta$ at $2\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots;$

that is, $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \ldots$ But $0 < \theta < \frac{\pi}{2}$.

The best θ is therefore $\frac{\pi}{4}$, or $45°$.