Arnoldo Gutierrez March 29, 2010

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Calculus Notes

Today: More Interesting Sequences (11.1)

Wednesday: Short HW9 due 5 pm

Thursday: Review

Monday: MT2 (bring a calculator & sheet of notes)

$a_0 = -1$

$a_1 = \sqrt{a_0+1} = \sqrt{-1+1} = \sqrt{0} = 0$

$a_2 = \sqrt{a_1+1} = \sqrt{0+1} = 1$

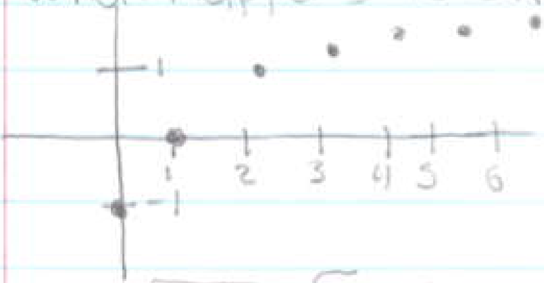
$a_3 = \sqrt{a_2+1} = \sqrt{1+1} = \sqrt{2} = 1.41421$

$a_4 = \sqrt{a_3+1} = \sqrt{\sqrt{2}+1} = \sqrt{1+\sqrt{2}} = 1.55377$

$a_5 = \sqrt{a_4+1} = \sqrt{\sqrt{1+\sqrt{2}}+1} = \sqrt{1+\sqrt{1+\sqrt{2}}} = 1.59805$

$a_6 = \sqrt{a_5+1} = \sqrt{\sqrt{1+\sqrt{1+\sqrt{2}}}+1} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} = 1.61185$

What happens to a_n when n is large?



Does this sequence keep increasing?

$3 \to \sqrt{3+1} = \sqrt{4} = 2$

Suppose, for the sake argument

$L = \lim_{n\to\infty} a_n$ exists

I claim $L = \sqrt{L+1}$

Suppose n is so large that a_n is very close to L. and $a_{n+1}, a_{n+2}, a_{n+3} \dots$ are all very close to L too.

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a_n is close to $L \Rightarrow \sqrt{a_n+1}$ is close $\sqrt{L+1}$

$a_{n+1} = \sqrt{a_n+1}$ is also close to L

Then L must be close to $\sqrt{L+1}$

Rigorous version

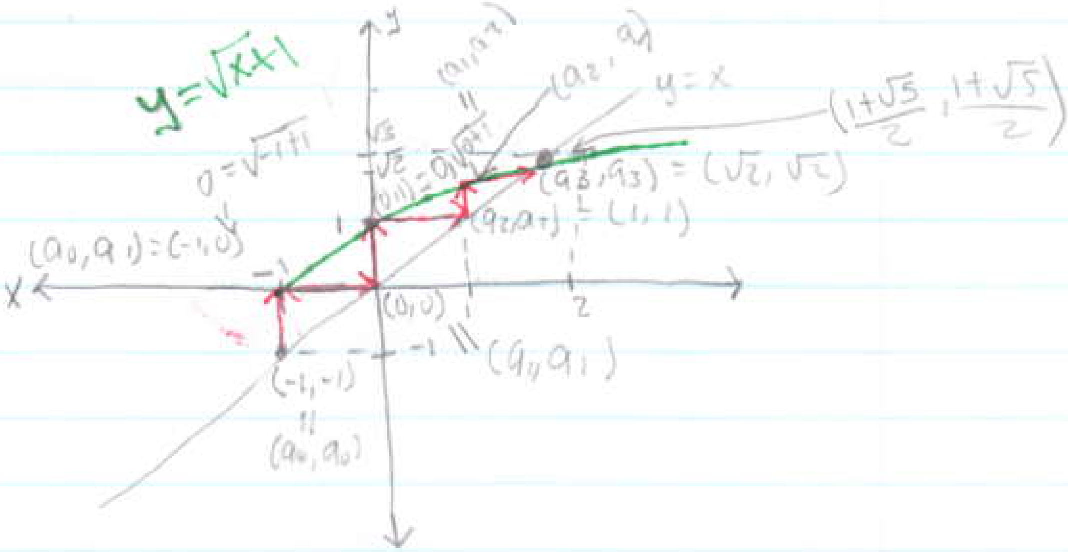
If $L = \lim_{n\to\infty} a_n$, then $L = \lim_{n\to\infty} a_{n+1}$

$\{a_n\}_{n=0}^{\infty}$ $\qquad$ $\{a_{n+1}\}_{n=0}^{\infty}$

$\{-1, 0, 1, \sqrt{2}, \sqrt{1+\sqrt{2}} \ldots\}$ $\qquad$ $\{0, 1, \sqrt{2}, \sqrt{1+\sqrt{2}} \ldots\}$

By continuity of $\sqrt{\ }$, $\boxed{\sqrt{L+1} = \lim_{n\to\infty} \sqrt{a_n+1} = \lim_{n\to\infty} a_{n+1} = L}$

our sequence is defined by $a_0 = -1$ & $a_{n+1} = \sqrt{a_n+1}$



limit point where curves intersect:

$y = x$ and $y = \sqrt{x+1}$

$x = \sqrt{x+1}$

$\downarrow$

Only solution: $x = \dfrac{1+\sqrt{5}}{2}$

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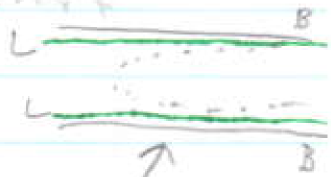
Proving $\lim_{n\to\infty} a_n$ exists...

we will use the "monotune covergence theorem", which says if a sequence is nondecreasing ($a_0 \le a_1 \le a_2 \le a_3 \le \dots$) or non increasing ($a_0 \ge a_1 \ge a_2 \ge a_3 \ge \dots$) and its bounded, meaning there is some bound B such that

$$\begin{cases} a_0 \le a_1 \le a_2 \le a_3 \le \dots \le B \\ \text{or } a_0 \ge a_1 \ge a_2 \ge a_3 \ge \dots \ge B, \end{cases}$$ then

$\lim_{n\to\infty} a_n$ exists.



$a_0 = -1$, $a_{n+1} = \sqrt{a_n + 1}$

$\{a_n\}_{n=0}^{\infty} = \{-1, 0, 1, \sqrt{2}, \sqrt{\sqrt{2}+1}, \sqrt{\sqrt{\sqrt{2}+1}+1}$ to $1, \dots\}$

I claim $\{a_n\}_{n=0}^{\infty}$ is nondecreasing and bounded

nondecreasing is easy to prove, if we prove a_n is always below $1+\sqrt{5}/2$

Remember $2 > \sqrt{2+1} = \sqrt{3}$. we want $a_n \le \sqrt{a_n+1} = a_{n+1}$

Suppose $-1 \le a_n < 1+\sqrt{5}/2$. Then we'll prove $-1 \le a_{n+1} < \frac{1+\sqrt{5}}{2}$ and $a_n \le a_{n+1}$

$a_n \le a_{n+1} \iff a_n \le \sqrt{a_n+1}$

$\Rightarrow a_n^2 \le a_n + 1 \Rightarrow a_n^2 - a_n - 1 \le 0$

Solve $a_n^2 - a_n - 1 = 0$: $a_n = \frac{1 \pm \sqrt{5}}{2}$

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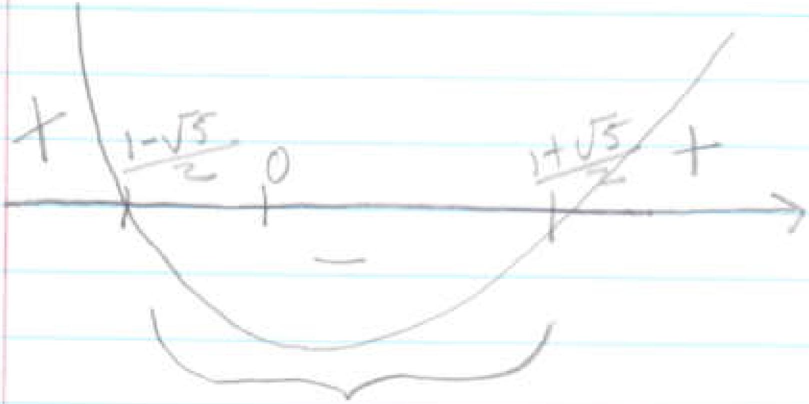

If we're in here, then $a_n^2 - a_n - 1 < 0$

except for the first term $a_0 = -1$, a_n is in $\left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$. Actually, a_n is in $\left[0, \frac{1+\sqrt{5}}{2}\right)$ after the first term.

$0 = a_1$ is in $\left[0, \frac{1+\sqrt{5}}{2}\right)$

If a_n is in $\left[0, \frac{1+\sqrt{5}}{2}\right)$, then so is a_{n+1}, so we never leave $\left[0, \frac{1+\sqrt{5}}{2}\right)$.

$0 \le a_n < \frac{1+\sqrt{5}}{2} \Rightarrow 1 \le a_n + 1 < \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2}$

$\Rightarrow 0 \le \sqrt{1} \le \sqrt{a_n + 1} < \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$

$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$

If a_n is in $\left[0, \frac{1+\sqrt{5}}{2}\right)$, then $a_n < \sqrt{a_n + 1} = a_{n+1}$

because $a_n^2 - a_n - 1 < 0$