March 31, 2010

Section 11.2 (continued)

Last time: $-1 < r < 1 \Rightarrow \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ (convergent)

What about other r?

If $\sum_{k=0}^{\infty} a_k$ converges and $= L$, then by definition,

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k = L, \text{ so also } \lim_{n \to \infty} \sum_{k=0}^{n+1} a_k = L \text{ too.}$$

$$\text{Therefore, } \lim_{n \to \infty} \underbrace{\left(\sum_{k=0}^{n+1} a_k - \sum_{k=0}^{n} a_k \right)}_{\underbrace{(a_0 + a_1 + \dots a_{n+1}) - (a_0 + a_1 + \dots a_n)}_{a_{n+1}}} = L - L = 0$$

$$\lim_{n \to \infty} a_{n+1} = 0$$

In a convergence series, eventually you're adding really small numbers.

Divergence Test:

If $\lim_{n \to \infty} a_n$ does not exist (as a finite number),

or $\lim_{n \to \infty} a_n = L \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.

$r = 1 \Rightarrow \lim_{m \to \infty} r^m = 1 \neq 0$

$r > 1 \Rightarrow \lim_{t \to \infty} r^t = \infty$ (divergeant) — Apply the divergence test

$r \leq -1 \Rightarrow \lim_{s \to \infty} r^s$ diverges — $\sum_{l=0} r^l$ diverges

-Example: $\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, 1, -1, 1 \ldots\}$

$\sum_{k=0}^{n} (-1)^k = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + \ldots (-1)^n$
$= 1 + -1 + 1 + -1 + \ldots (-1)^n$

$(-1)^n = 1$ if n even ; $(-1)^n = -1$ if n odd

0 if n odd, 1 if n even $\Rightarrow \sum_{k=0}^{\infty} (-1)^n$ divergent

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-Example:

$$\sum_{j=0}^{\infty} j = 0 + 1 + 2 + 3 + 4 + 5 + \ldots = \infty$$

$$\lim_{j \to \infty} j = \infty$$

-Example:

$$\sum_{l=1}^{\infty} \left(3 + \frac{1}{l}\right) = \left(3 + \frac{1}{1}\right) + \left(3 + \frac{1}{2}\right) + \left(3 + \frac{1}{3}\right) + \ldots \infty$$

$$\lim_{l \to \infty} \left(3 + \frac{1}{l}\right) = 3 + 0 = 3 \neq 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = ? \qquad \text{(Harmonic series)}$$

$\lim_{n\to\infty} \frac{1}{n} = 0$, so the divergence test doesn't say anything.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots = ?$$

$\frac{1}{1} = 1$

$\frac{1}{1} + \frac{1}{2} = 1.5$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \approx 1.833$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.083$

$\frac{1}{1} + \ldots \frac{1}{5} \approx 2.283$

$\frac{1}{1} + \ldots \frac{1}{6} \approx 2.45$

$\frac{1}{1} + \ldots \frac{1}{7} \approx 2.593$

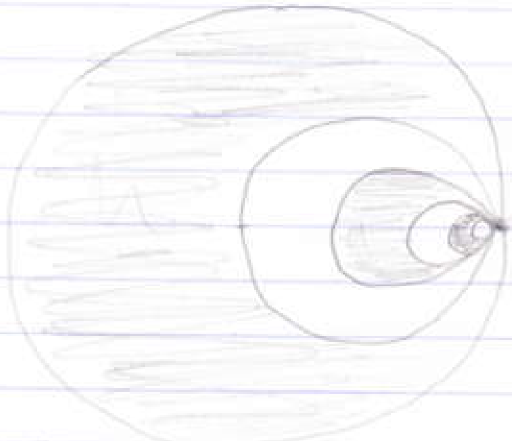
$\frac{1}{1} + \ldots \frac{1}{8} \approx 2.718$

$\frac{1}{1} + \ldots \frac{1}{20} \approx 3.723$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{16} \ldots$$

$$\geq \underbrace{\frac{1}{1}}_{1} + \underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2} + \underbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}}_{1/2} \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \infty} \text{ divergent}$$



Infinite hoop earing.

Start with a circle of radius $r_0 = 50cm$

Subtract a circle with radius $r_1 = 25cm = \frac{50}{2}cm = \frac{r_0}{2} = \frac{r_0}{2^1}$

Add a circle with radius $r_2 = 12.5cm = \frac{50}{4}cm = \frac{r_0}{4} = \frac{r_0}{2^2}$

Subtract a circle with radius $r_3 = 6.25cm = \frac{50}{8} = \frac{r_0}{8} = \frac{r_0}{2^3}$

Repeat ad infinitum $r_n = \frac{r_0}{2^n}$

n^{th} area : $A_n = \pi r_n^2 = \pi \left(\frac{r_0}{2^n}\right)^2$

$$A_n = \frac{\pi r_0^2}{4^n} = \pi r_0^2 \left(\frac{1}{4}\right)^n$$

A = Final Area $= A_0 - A_1 + A_2 - A_3 + A_4 - A_5 \ldots = ?$

$$A = \sum_{n=0}^{\infty} (-1)^n A_n = \sum_{n=0}^{\infty} (-1)^n \pi r_0^2 \left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty} \pi r_0^2 \left(-\frac{1}{4}\right)^n \Rightarrow$$

$$\Rightarrow \underbrace{\pi r_0^2}_{\text{constant}} \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \pi r_0^2 \left(\frac{1}{1-\left(-\frac{1}{4}\right)}\right) = \pi r_0^2 \left(\frac{1}{5/4}\right) = \pi r_0^2 \left(\frac{4}{5}\right)$$

$$= \boxed{2000 \pi \, cm^2}$$

$\sum_{n=0}^{\infty} \frac{2^n}{n!} = ?$ → This converges (to e^2)

$0! = 1$ (by definition)
$1! = 1$
$2! = 1 \cdot 2 = 2$
$3! = 1 \cdot 2 \cdot 3 = 6$
$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$
$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cancel{5} \cdot 6 = 720$

$\sum_{n=0}^{\infty} \frac{r^n}{n!}$ converges for all "r"

$\sum_{n=0}^{\infty} \frac{n!}{r^n}$ diverges for all $r \neq 0$

↳ because:

$\lim_{n \to \infty} \frac{n!}{r^n} = \infty$ if $r \neq 0$

$$\frac{n!}{r^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots\cdots (n-2)(n-1)(n)}{r \cdot r \cdot r \cdot r \cdot r \cdot r \cdots\cdots r \cdot r \cdot r}$$

after $n > r$, this keeps getting bigger, getting as big as you want.