CALCULUS II 04/05

11.3 Integral Test

(Tomorrow: MT2, Bring calculator & 1 sheet of notes)

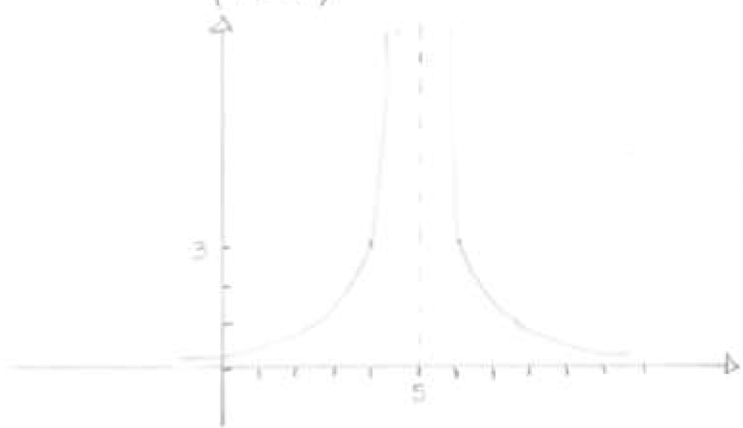
$$\sum_{n=7}^{\infty} \frac{3}{(n-5)^2} = \frac{3}{(7-5)^2} + \frac{3}{(8-5)^2} + \frac{3}{(9-5)^2} + \frac{3}{(10-5)^2} + \dots = \frac{3}{4} + \frac{3}{9} + \frac{3}{16} + \dots$$

Is it convergent or divergent?

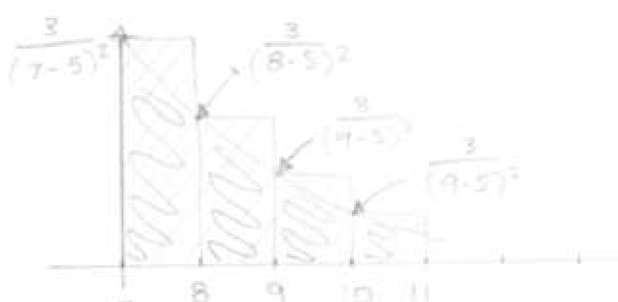
n big $\Rightarrow \frac{3}{(n-5)^2}$ small $\Rightarrow \lim_{n \to \infty} \frac{3}{(n-5)^2} = 0$

Divergence test says nothing

Graph $y = \frac{3}{(n-5)^2}$



Use left-endpoint rule with $\Delta x = 1$ or $[7, \infty)$



▭ Area $= \frac{3}{(7-5)^2} + \frac{3}{(8-5)^2} + \frac{3}{(9-5)^2} + \dots = \sum_{n=7}^{\infty} \frac{3}{(n-5)^2}$

▭ Area $= \int_7^{\infty} \frac{3\,dx}{(x-5)^2}$ Because $\frac{3}{(x-5)^2}$ $\underbrace{\text{slopes down}}_{\text{decreasing}}$ and is positive

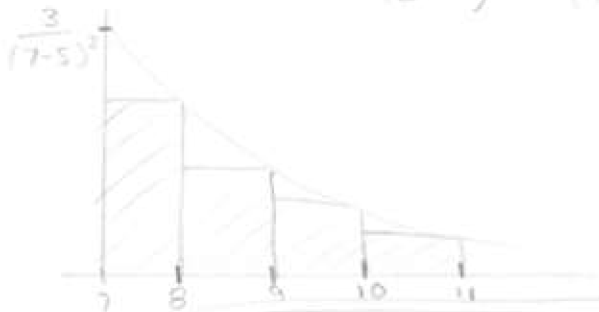
$$\sum_{n=7}^{\infty} \frac{3}{(n-5)^2} \geq \int_7^{\infty} \frac{3\,dx}{(x-5)^2} = \int_2^{\infty} \frac{3\,du}{u^2} \quad \left.\right\} \begin{array}{l} u = x-5 \\ du = dx \end{array}$$

$$\lim_{t \to \infty} \int_2^t 3u^{-2}\,du = \int_2^{\infty} 3u^{-2}\,du = \lim_{t \to \infty} 3 \cdot \left. \frac{u^{-2+1}}{-2+1} \right|_2^t =$$

$$= -3 \lim \left. u^{-1} \right|_2^t = -3 \lim_{t \to \infty} \left(t^{-1} - 2^{-1}\right) = \frac{3}{2} \Rightarrow \boxed{\sum_{n=7}^{\infty} \frac{3}{(n-5)^2} \geq \frac{3}{2}}$$

Use right-endpoint too

$$\text{Area} = \frac{3}{(8-5)^2} + \frac{3}{(9-5)^2} + \frac{3}{(10-5)^2} + \dots = \sum_{n=8}^{\infty} \frac{3}{(n-5)^2}$$



Because

$$\int_7^{\infty} \frac{3\,dx}{(x-5)^2} \geq \sum_{n=8}^{\infty} \frac{3}{(n-5)^2} = \frac{3}{9} + \frac{3}{16} + \frac{3}{25} \dots$$

$$\int_7^{\infty} \frac{3\,dx}{(x-5)^2} = \frac{3}{2} \Rightarrow \sum_{n=7}^{\infty} \frac{3}{(n-5)^2} = \sum_{n=8}^{\infty} \frac{3}{(n-5)^2} + \frac{3}{(7-5)^2} \leq \int_7^{\infty} \frac{3\,dx}{(x-5)^2} + \frac{3}{(7-5)^2}$$

$$\frac{3}{2} + \frac{3}{4} = \boxed{\frac{9}{4}}$$

$$\int_7^{\infty} \frac{3\,dx}{(x-5)^2} = \underset{\substack{\parallel \\ 1.5}}{\frac{3}{2}} \leq \sum_{n=7}^{\infty} \frac{3}{(n-5)^2} \leq \underset{\substack{\parallel \\ 2.25}}{\frac{9}{4}} = \int_7^{\infty} \frac{3\,dx}{(x-5)^2} + \frac{3}{(7-5)^2}$$

$$\sum_{n=7}^{\infty} \frac{3}{(n-5)^2} = \frac{\pi^2}{2} - 3 \approx 1.9348$$

(more advanced than calculus)

$$\frac{3}{2^2} + \frac{3}{3^2} + \frac{3}{4^2} + \dots + \frac{3}{17^2} \approx 1.76342$$

Try on your own

$$\frac{3}{2^2} + \frac{3}{3^2} + \frac{3}{4^2} + \dots + \frac{3}{1000^2} \approx ?$$

Integral test: If k is integer, and $f(x)$ is continuous, decreasing ($\searrow$), and positive on $[k, \infty)$, then

$$\int_k^\infty f(x)\,dx \text{ divergent} \Rightarrow \sum_{n=k}^{\infty} f(n) \text{ divergent}$$

$$\int_k^\infty f(x)\,dx \text{ convergent} \Rightarrow \sum_{n=k}^{\infty} f(n) \text{ convergent}$$

Moreover:

$$\int_k^\infty f(x)\,dx \le \sum_{n=k}^{\infty} f(n) \le \int_k^\infty f(x)\,dx + f(k)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

Why? Integral test:

$$\int_1^\infty \frac{dx}{x^p} \le \sum_{n=1}^{\infty} \frac{1}{n^p} \le \int_1^\infty \frac{dx}{x^p} + \frac{1}{1^p}$$

$$\lim_{t \to \infty} \int_1^t x^{-p}\,dx = \lim_{t \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^t = \lim_{t \to \infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} \right)$$

$$= \begin{cases} \infty : -p+1 > 0 \\ 0 - \frac{1}{-p+1} : -p+1 < 0 \end{cases}$$

$\text{big}^{(+\text{constant})} = \text{big}$ $\quad 1000\,000^{1/2} = 1000$

$\text{big}^{(-\text{constant})} = \text{small}$ $\quad 1000\,000^{-1/3} = \frac{1}{100}$

$$p = 1 \Rightarrow -p+1 = 0 \Rightarrow \lim_{t \to \infty} \int_1^t x^{-1}\,dx = \lim_{t \to \infty} \ln|x| \Big|_1^t$$

$$\Rightarrow \lim_{t \to \infty} (\ln(t) - 0) = \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \Longleftrightarrow p>1 \Rightarrow -p+1<0 \Rightarrow \int_1^{\infty} x^p\,dx = \frac{1}{1-p} \\ \text{diverges if } p \leq 1 \Longleftrightarrow \begin{cases} p<1 \Rightarrow -p>-1 \Rightarrow -p+1>0 \Rightarrow \int_1^{\infty} x^p\,dx = \infty \\ p=1 \Rightarrow \int_1^{\infty} x^p\,dx = \infty \end{cases} \end{cases}$$