Eddie

4/20 Today: 11.7: Review of convergence of series

Tmwk: series of polynomials (11.8)

Next Monday: Hmwk 12 due @ 5

[pg 1]

"Fundamental" Series

① $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ← constant "p-series" $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \le 1 \end{cases}$

---

② $\sum_{n=0}^{\infty} r^n = \begin{cases} 1/(1-r) : -1<r<1 \text{ (same as } |r|<1) \\ \text{diverges : otherwise (same as } |r| \ge 1) \end{cases}$

"geometric series"

The Convergence Tests

- Comparison tes
- Limit comparison test
- Divergence test
- Alternating Series test
- Ratio test
- Root test
- Integral test

$\sum_{n=3}^{\infty} \frac{1}{n^2+5}$

$0 \le \frac{1}{n^2+5} \le \frac{1}{n^2}$

$0 = \sum_{n=3}^{\infty} 0 \le \sum_{n=3}^{\infty} \frac{1}{n^2+5} \le \underbrace{\sum_{n=3}^{\infty} \frac{1}{n^2}}_{\substack{\text{p-series} \\ p=2>1}} < \infty$

By the comparing test,

$\sum_{n=3}^{\infty} \frac{1}{n^2+5}$ converges

Example: $\sum_{n=2}^{\infty} \frac{\sqrt{3n^5+n^{7/4}}}{n^3+1}$ → compare to $\sum_{n=2}^{\infty} \frac{\sqrt{n^5}}{n^3} \Rightarrow \sum_{n=2}^{\infty} \frac{n^{5/2}}{n^3} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^{1/2}} \Rightarrow \underbrace{\infty}_{\text{diverges}}$

p-series

$p = \frac{1}{2} \le 1$

Continued on next pg

Use limit Comparison test:

$a_n = \frac{\sqrt{3n^5 + n^{7/4}}}{n^3+1} \qquad b_n = \frac{\sqrt{n^5}}{n^3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{3n^5+n^{7/4}}/\sqrt{n^5}}{(n^3+1)/n^3} = \lim_{n\to\infty} \frac{\sqrt{3+n^{7/4-5}}}{1+1/n^3} \Rightarrow \lim_{n\to\infty} \frac{\sqrt{3+1/n^{13/4}}}{1+1/n^3}$

$\Downarrow$

Eventually $\frac{a_n}{b_n} > \sqrt{3} - 0.001 \leftarrow \sqrt{3} \leftarrow \frac{\sqrt{3+0}}{1+0}$

$\Downarrow$

Eventually $a_n > (\sqrt{3} - 0.001) b_n$

Since $\sum_{n=2}^{\infty} b_n$ diverges, $\sum_{n=2}^{\infty} a_n$ diverges too!

So... by the LCT, $\sum_{n=2}^{\infty} a_n = \infty$

In general, if

$L = \lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite non-zero number, then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$

either both converge or both diverge. ← section 11.4

← need a_n & $b_n > 0$.

Other tricks for comparisons: $n \geq 1 \Rightarrow 0 \leq \ln(n) < n$

for all n, $-1 \leq \sin(n) \leq 1$

Divergence test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

$\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$ diverges because: $\lim_{n\to\infty} n \sin(\frac{1}{n}) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to 0} \frac{\sin x}{x} \, {}^{\to 0}_{\to 0}$

$= \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1 \neq 0$

- If $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n$ may or may not converge.

Easy $\sum_{n=5}^{\infty} n^2 = \infty$ because $\lim_{n\to\infty} n^2 = \infty \neq 0$

$\Downarrow$

- Ratio test: good for products of n things like 5^n, $n!$, n^n...

- Root test: good for n^{th} powers of variable things, like: n^n, $\left(\frac{1}{\ln(n)}\right)^n$, $\left(\left(1-\frac{1}{n}\right)^n\right)^n$, $(\cos(3n))^n$, ...

- Ratio + Root test are bad for things like $\frac{n^3}{n^5-7n}$ ← Use LCT instead

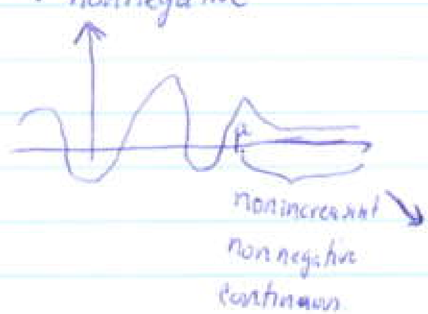
- Alternating Series Test: good for:

$(-1)^n b_n$ or
$(-1)^{n+1} b_n$ or
$(-1)^{n-1} b_n$, ...

If $b_0 \geq b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0$, then

$$\begin{cases} \lim_{n\to\infty} b_n = 0 \Rightarrow \sum_{n=0}^{\infty} (-1)^n b_n \text{ converges} \\ \lim_{n\to\infty} b_n \neq 0 \Rightarrow \lim_{n\to\infty} (-1)^n b_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} (-1)^n b_n \text{ diverges} \end{cases}$$

- Integral test: If $f(x)$ is eventually non increasing + nonnegative and eventually continuous, say on $[a,\infty)$, then



$$\int_a^\infty f(x)\,dx < \infty \Rightarrow \sum_{n=0}^{\infty} f(n) \text{ converges}$$

$$\int_a^\infty f(x)\,dx = \infty \Rightarrow \sum_{n=0}^{\infty} f(n) \text{ diverges}$$

Little example → $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ $\quad f(x)\ \frac{1}{x\ln^2 x}$ $\quad x>1 \Rightarrow x + \ln x > 0$

x, x^2, $\ln x$ are ↗ everywhere

so $x\cdot(\ln x)^2$ is ↗ everywhere

so $\boxed{\frac{1}{x(\ln x)^2} \text{ is } \searrow \text{ on } (1,\infty).}$ also cts on $(1,\infty)$ b/c x + $\ln x$ are contin. + $\neq 0$ on $(1,\infty)$

$$\int_2^\infty \frac{dx}{x\ln^2 x} = \int_{\ln 2}^\infty \frac{du}{u^2} = \frac{1}{\ln 2} < \infty$$

Pg 4

$u = \ln x \qquad x = 2 \Rightarrow u = \ln 2$

$\qquad\qquad\quad x \to \infty \Rightarrow u \to \infty$

$\frac{dy}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$

$$\int_a^\infty \frac{dx}{x^p} \begin{cases} \text{converges} : p > 1 \\ \text{diverges} : p \leq 1 \end{cases}$$

$a > 0$

by the IT, $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ converges