Wednesday April 21, 2010.

Today: Power Series (11.8)
Tommorrow: 11.9?

$$1+x+x^2+x^3+x^4+x^5+x^6+\ldots=\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} : & -1<x<1 \\ \text{diverges} : & \text{otherwise} \end{cases}$$

geometric series

For which $x$ does this converge?

all $x$ in $(-1,1)$

For which $x$ does $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$ converge?

Let's agree that $x^0=1$ for all $x$ so $0!=1$

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad b_n=\frac{x^n}{n!}$$

Ratio test

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \frac{|x|^{n+1}\, n!}{|x|^n (n+1)!} = \frac{|x|\, n!}{(n+1)!}$$

$$= \frac{|x|\ 1\cdot 2\cdot 3\cdot 4\ldots(n-1)\cdot n}{1\cdot 2\cdot 3\cdot 4\ldots(n-1)\cdot n\cdot(n+1)} = \frac{|x|}{n+1}$$

$$\lim_{n\to\infty}\frac{|x|}{n+1} = |x|\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0<1 \Rightarrow \text{converges for any } x.$$

$$\sum_{n=0}^{\infty}(3+1/n)^n x^n$$

For which $x$ does this converge?

Root Test: $\quad b_n=(3+1/n)^n x^n$

$$\sqrt[n]{|b_n|} = \left(3+\frac{1}{n}\right)|x| \Rightarrow \lim_{n\to\infty}\sqrt[n]{|b_n|} = (3+0)|x| = 3|x|$$

$3|x| < 1 \Rightarrow S$ converges

$3|x| > 1 \Rightarrow S$ diverges

$3|x| = 1 \Rightarrow$ root test is inconclusive

$3|x| < 1$ ✓

$\Leftrightarrow |x| < \frac{1}{3}$

$\Leftrightarrow -\frac{1}{3} < x < \frac{1}{3}$

$\Leftrightarrow x$ in $\left(-\frac{1}{3}, \frac{1}{3}\right)$

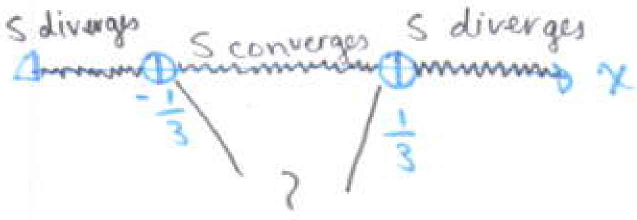
$\underbrace{\uparrow \text{ S converges here.}}$

$3|x| > 1 \Leftrightarrow |x| > \frac{1}{3}$

$\Leftrightarrow x < -\frac{1}{3}$ or $\frac{1}{3} < x$

$\Leftrightarrow x$ in $\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

↑ S diverges here.




Look at $\sum_{n=0}^{\infty} \left(3 + \frac{1}{n}\right)^n \left(-\frac{1}{3}\right)^n$ and $\sum_{n=0}^{\infty} \left(3 + \frac{1}{n}\right)^n \left(\frac{1}{3}\right)^n \ldots$

You can check that both diverge:

Use the divergence test:

$$\lim_{n \to \infty} \left(3 + \frac{1}{n}\right)^n \left(\pm \frac{1}{3}\right)^n = e^{\pm 1/3} \neq 0$$

Why do we care?

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ for all $x$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$ for all $x$ in $(-1, 1)$

$$\int_0^{1/2} \frac{dx}{1-x} = \int_0^{1/2} (1+x+x^2+x^3+x^4+\dots)\,dx$$

$u = 1-x$ $\quad x=0 \Rightarrow u=1$

$du = -dx$ $\quad x=1/2 \Rightarrow u = 1/2$

$$\int_1^{1/2} -\frac{du}{u} = x\Big|_0^{1/2} + \frac{x^2}{2}\Big|_0^{1/2} + \frac{x^3}{3}\Big|_0^{1/2} + \frac{x^4}{4}\Big|_0^{1/2} + \frac{x^5}{5}\Big|_0^{1/2} + \dots$$

$$-\ln|u|\Big|_1^{1/2} = \left(\frac{1}{2}\right) + \left(\frac{1/4-0}{2}\right) + \left(\frac{1/8-0}{3}\right) + \left(\frac{1/8-0}{4}\right) + \left(\frac{1/32-0}{5}\right) + \dots$$

$$\underbrace{-\ln\frac{1}{2}}_{\ln\frac{1}{1/2}} - \underbrace{(-\ln 1)}_{0} = \underbrace{\frac{1}{1\cdot 2}}_{n=1} + \underbrace{\frac{1}{2\cdot 4}}_{n=2} + \underbrace{\frac{1}{3\cdot 8}}_{n=3} + \underbrace{\frac{1}{4\cdot 16}}_{n=4} + \underbrace{\frac{1}{5\cdot 32}}_{n=5} + \dots + \underbrace{\frac{1}{n\cdot 2^n}}_{\dots}$$

$$\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n\cdot 2^n} \qquad \ln 2 = 0.6931471805 9995\dots$$

$$\frac{1}{1\cdot 2} = 0.5$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} = \frac{5}{8} = 0.625$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} = \frac{2}{3} = 0.66666\dots$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \frac{1}{4\cdot 16} = \frac{131}{192} = 0.682292\dots$$

$$\frac{1}{1\cdot 2} + \dots + \frac{1}{5\cdot 32} = 0.688542\dots$$

$$\frac{1}{1\cdot 2} + \dots + \frac{1}{10\cdot 2^{10}} = 0.693065\dots$$

$$\frac{1}{1\cdot 2} + \dots + \frac{1}{20\cdot 2^{20}} = 0.69314713\dots$$

$$\text{error} = \ln 2 - \sum_{n=1}^{20} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} - \sum_{n=1}^{20} \frac{1}{n \cdot 2^n}$$

$$= \sum_{n=21}^{\infty} \frac{1}{n \cdot 2^n} < \sum_{n=21}^{\infty} \frac{1}{2^n} = \sum_{m=0}^{\infty} \frac{1}{2^{m+21}} = \frac{1}{2^{21}} \underbrace{\sum_{m=0}^{\infty} \frac{1}{2^m}}_{\text{geometric series.}}$$

$m = n - 21 \Rightarrow n = m + 21$

$n = 21 \Rightarrow m = 0$

$n \to \infty \Rightarrow m \to \infty$

$$\text{error} < \frac{1}{2^{21}} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = \frac{1}{2^{21}} \left(\frac{1}{1 - \frac{1}{2}}\right) = \frac{1}{2^{21}} \left(\frac{1}{\frac{1}{2}}\right) = \frac{1}{2^{20}} < \frac{1}{10^6}.$$

$$\ln 2 - \sum_{n=1}^{k} \frac{1}{n \cdot 2^n} < \frac{1}{2^k}$$