11.8/11.9 (Power Series Today)

Mondays $\begin{cases} 11.9/11.10 \\ \text{HW Due} \end{cases}$

If $c_0, c_1, c_2, c_3, c_4 \ldots\ldots$ are (real) numbers, a is

a (real) number, then

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 \ldots,$$

is a power series

The simpler case $a=0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\cdots$$

is a power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1 x^n = 1 + x + x^2 + x^3 \cdots\cdots$$

converges if $|x| < 1$

Every power series, say $\sum_{n=0}^{\infty} c_n (x-a)^n$

has a radius of convergence R

$R=0$ means $\sum_{n=0}^{\infty} c_n (x-a)^n$ only converges if

$x-a=0$

Example: $\sum_{n=0}^{\infty} n!(x-2)^n$ only converges if $x-2=0$

$R=\infty$ means $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all x

Example $\sum_{n=0}^{\infty} \frac{(x+3)^n}{n!}$ converges (to e^{x+3}) for all x

$1!=1 \quad 2!=1\cdot 2=2 \quad 3!=1\cdot 2\cdot 3=6 \ldots$

$0<R<\infty$ means: $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges if $|x-a|<R$

(same as x in $(a-R, a+R)$)

$\sum_{n=0}^{\infty} c_n(x-a)^n$ diverges if $|x-a|>R$ (same as x $(-\infty, a-r)$ $\cup (a+R,\infty)$)

If $x=a\pm R$, then you have to do more work to determine if $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges or diverges.

Example: $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ $= \ln(6-x)$

$$\begin{cases} \text{converges if } |x-5|<1 \\ \text{diverges if } |x-5|>1 \\ \text{converges if } x=5-1=4 \\ \text{diverges if } x=5+1=6 \end{cases}$$

Find R using ratio or root test.

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \sqrt[n]{c_n}$ (if they truly exist)

with the understanding that $\begin{cases} \frac{1}{R}=\infty \text{ means } R=0 \\ \frac{1}{R}=0 \text{ means } R=\infty \end{cases}$

Then $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence R.

$\sum_{n=0}^{\infty} \frac{n^2 x^n}{4^n}$

Ratio Test : $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2/n^2}{4^{n+1}/4^n} = \lim_{n\to\infty} \frac{((n+1)/n)^2}{4}$

$/ \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{4} = \frac{(1+0)^2}{4} = \frac{1}{4} = R = 4 \quad c_n = \frac{n^2}{4^n}$

$\sum_{n=0}^{\infty} \frac{n^2 x^4}{4^n} \begin{cases} \text{converges} & \text{if } |x| < 4 \\ \text{diverges} & \text{if } |x| > 4 \end{cases}$

Why does this work $\frac{1}{R} = \lim \cdots$

$a_n = c_n x^n = \frac{n^2 x^n}{4^n}$

The ratio test $\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & L < 1 \\ \text{diverges} & L > 1 \end{cases}$

where $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1} x^{n+1}}{c_n x^n}\right|$

$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| |x| = \left(\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|\right) |x| = \frac{1}{R} (x)$

converges if $\frac{1}{R} x < 1 \quad |x| < R$
diverges if $\frac{1}{R} x > 1 \quad |x| > R$

Let $F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Let R be the radius of convergence.

(Note that R could be $0, \infty$ or in between)

For all x in $(a-R, a+R)$, all derivatives & antiderivatives of f exist and they can be computed "term by term"

$$f'(x) = \sum_{n=0}^{\infty} (c_n (x-a)^n)' = \sum_{n=0}^{\infty} c_n \underbrace{n}_{=0 \text{ when } n=0} (x-a)^{n-1} = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}$$

$$\sum_{m=0}^{\infty} c_{m+1} (m+1) (x-a)^m \quad m=n-1$$

$$\int f(x)\,dx = \left(\sum_{n=0}^{\infty} \int c_n (x-a)^n\,dx\right) + C_{-1}$$

$$\sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C_{-1} \qquad a-R < P \le Q < a+R$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \int_P^Q c_n (x-a)^n\,dx$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = S$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(-1)^n}\right| = \lim_{n\to\infty} \frac{1}{1} = 1 \Rightarrow R = 1 \Rightarrow S$$ converges for all x

in $(-1, 1)$

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^2} = \cancel{\tan^{-1}x \Big|_0^{1/\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}}$$

$$\sum_{n=0}^{\infty} (-1)^n \int_0^{\frac{1}{\sqrt{3}}} x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{2n+1}\right)\Big|_0^{1/\sqrt{3}}$$

$$\tan^{-1}x \Big|_0^{1/\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1} - 0}{2n+1} \Rightarrow \pi = 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}$$

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}\cdots\right)$$