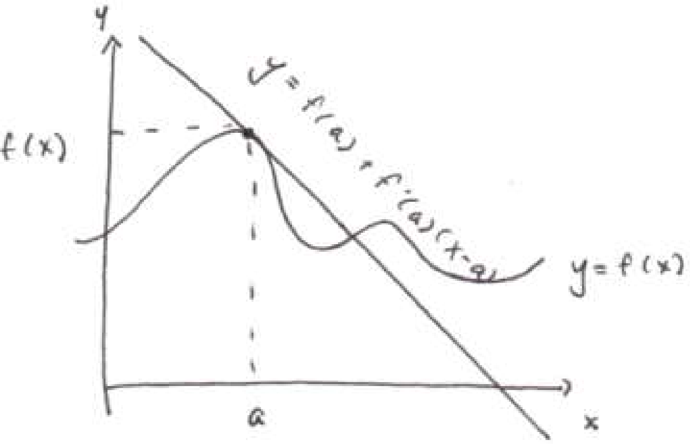
Chapter 11 Sec. 10

Taylor Series



Tangent line Approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

for $x \approx a$

Parabola approximation

$$f(x) \approx f(a) + f'(a)(x-a) + (???)(x-a)^2$$

for $x \approx a$ $\qquad$ $\frac{f''(a)}{2}$

$$g(x) = ax^2 + bx + c = \frac{g''(0)}{2}(x-0)^2 + g'(0)(x-0) + g(0)$$

$g(0) = c$

$g'(x) = 2ax + b$

$g'(0) = b$

$g''(x) = 2a$

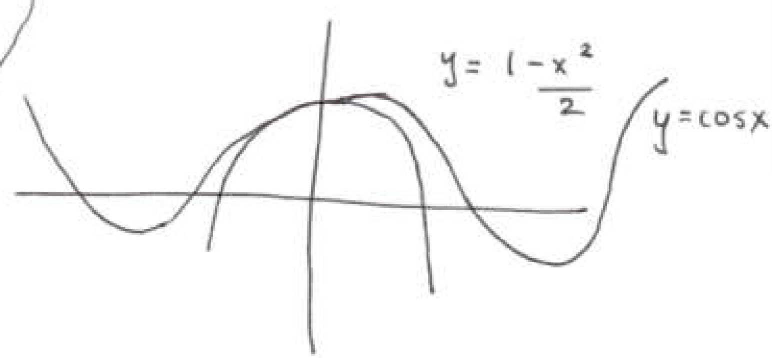
$g''(0) = 2a$

$$\cos x \approx \cos 0 + (\cos' 0)(x-0) + \frac{(\cos'' 0)}{2}(x-0)^2 = 1 - \frac{x^2}{2}$$

$\cos 0 = 1$

$\cos' 0 = -\sin 0 = 0$

$\cos'' 0 = -\sin 0 - \cos 0 = -1$



Next cubic approximations

$$h(x) = ax^3 + bx^2 + cx + d$$

$$h'(x) = 3ax^2 + 2bx + c$$

$$h''(x) = 6ax + 2b$$

$$h'''(x) = 6a$$

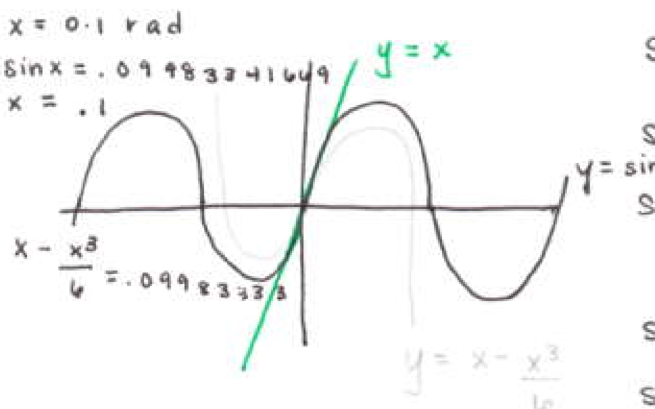
$$\left.\begin{array}{l} h(0) = d \\ h'(0) = c \\ h''(0) = 2b \\ h'''(0) = 6a \end{array}\right\} \Rightarrow h(x) = \underbrace{d + cx + bx^2 + ax^3}_{\displaystyle \parallel}$$

$$h(0) + h'(0)x + \frac{h''(0)}{2}x^2 + \frac{h'''(0)x^3}{6}$$

$$f(x) \approx f(a) + f'(a)(x-a) + \left(\frac{f''(a)}{2}\right)(x-a)^2 + \left(\frac{f'''(a)}{6}\right)(x-a)^3$$

for $x \approx 0$

Try $a = 0$; $f(x) = \sin x$



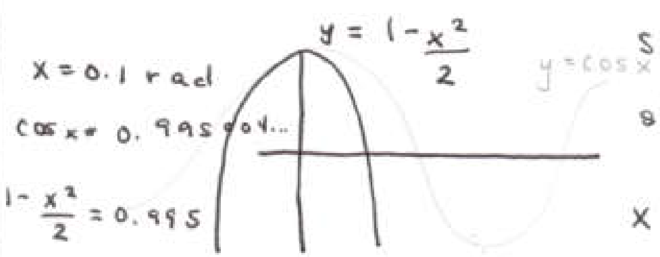
$\sin' x = \cos x$

$\sin'' x = \cos' x = -\sin x$

$\sin''' x = (\sin'' x)' = -\sin x = -\cos x$

$\sin 0 = 0$

$\sin' 0 = \cos 0 = 1$



$\sin'' 0 = -\sin 0 = 0$

$\sin''' 0 = -\cos 0 = -1$

$$x \approx 0 \Rightarrow \sin x \approx 0 + 1(x-0) + \frac{0}{2}(x-0)^2 + \frac{-1}{6}(x-0)^3$$

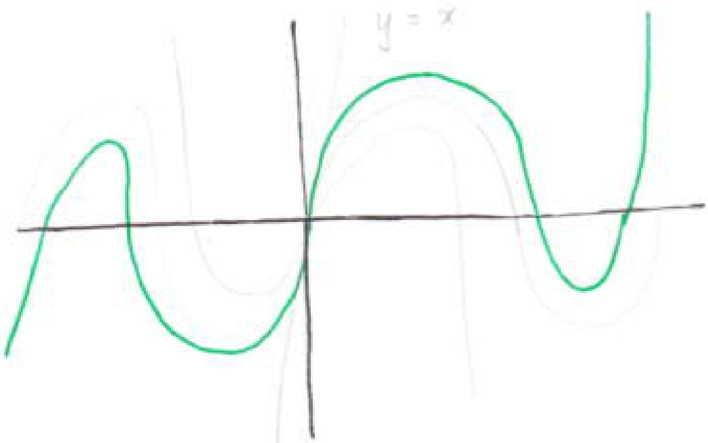
$$= x - \frac{x^3}{6}$$

The pattern: $x \approx a$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5$$

You can add more terms to get a better approximation.

Ex.



$$y = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$x = .1$

$$x - \frac{x^3}{6} + \frac{x^5}{120} = 0.9983341666..$$

For "nice" functions (include $\ln x$, e^x, $\sin x$, $\cos x$),

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

for all x in $(a-R, a+R)$, where R is the radius of convergence.

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \sqrt[n]{|c_n|}$$

where $c_n = \dfrac{f^{(n)}(a)}{n!}$

Ex. $(e^x)^{(n)} = e^x$ because $(e^x)' = e^x$

$\uparrow$ for all n

$(e^x)^{(n)}$ at $x=0$ is e^0, which is 1.

$$e^x = \sum_{n=0}^{\infty} \frac{1 \cdot (x-0)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$c_n = \frac{1}{n!} \Rightarrow \frac{c_{n+1}}{c_n} = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \Rightarrow 0$$

as $n \to \infty$

$\Rightarrow \frac{1}{R}$ " $=$ " $0 \Rightarrow R = \infty \Rightarrow$ This series converges everywhere

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$$

$\uparrow$ for all x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$\uparrow$ for all x $\downarrow$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Estimating $\sin(.1)$:

$a = .1 - \frac{.1^3}{3!}$ $\qquad$ $b = .1 - \frac{.1^3}{3!} + \frac{.1^5}{5!}$

$$\sin(.1) = \frac{a+b}{2} + \text{error}$$

$$|\text{error}| \le \frac{1}{2}(b-a) = \frac{1}{2} \frac{.1^5}{5!} = \frac{.00001}{240}$$

We use alternating series estimations when we can. For a positive sequence of terms, like

$$e = e^1 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

you need a different technique for guaranteeing small error.

By repeatedly integrating by parts, you can prove that:

$$f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \dots + \frac{f^{(K)}(a)}{K!}(x-a)^K$$

$$+ \underbrace{R_K(x)}_{\text{the error}}$$

$$R_K(x) = \int_a^x \frac{f^{(K+1)}(t)}{K!}(x-t)^K \, dt$$

If $|f^{(K+1)}(t)| \leq M$ for all t between a & x, then

$$|R_K(x)| \leq \frac{M|(x-a)|^{K+1}}{(K+1)!}$$