 4/29/10

Today 11.10/11.11

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n \quad \text{Taylor Series}$$

Nicer special case:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n \quad \text{Maclaurin series}$$

These equations are true most of the time, for "nice" functions.

A non-nice function:

$$f(x) \begin{cases} e^{-1/x} & : x > 0 \\ 0 & : x \leq 0 \end{cases} \Rightarrow f^{(n)}(0) = 0 \Rightarrow \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}_{\text{for all } n} \;\; \underset{\substack{\| \\ 0 \neq f(x)}}{}$$

$f(x)$	Maclaurin series	R = radius of convergence
e^x	$1 + x + x^2/2! + x^3/3! + \ldots$	∞
$\sin x$	$x - x^3/3! + x^5/5! - \ldots$	∞
$\cos x$	$1 - x^2/2! + x^4/4! - \ldots$	∞
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \ldots$	1
$\ln(1+x)$	$x - x^2/2 + x^3/3 - x^4/4 \ldots$	1
$\tan^{-1} x$	$x - x^3/3 + x^5/5 \ldots$	1
$(1+x)^a$	$1 + ax + \frac{a(a-1)}{2!} + \frac{a(a-1)(a-2)}{3!}$	1

Binomial Series
$x \approx 0 \Rightarrow (1+x)^a \approx 1 + ax + \frac{a(a-1)}{2} x^2$

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \frac{1}{1-(-t)} dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - \ldots) dt$$

$$= \int_0^x 1 \, dt - \int_0^x t \, dt + \int_0^x t^2 dt \ldots = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \ldots$$

$\Longrightarrow$

(2)

$f(x)=\ln(1+x)$ $\quad f(0)=\ln 1=0$ $\quad f'''(x)=-(-2)(1+x)^{-3}(1)$

$f'(x)=\frac{1}{1+x}$ $\quad f'(0)=\frac{1}{1+0}=1$ $\quad =\frac{2}{(1+x)^3}$

$f''(x)=\frac{-1}{(1+x)^2}$ $\quad f''(0)=\frac{-1}{(1+0)^2}=-1$ $\quad f'''(0)=\frac{2}{(1+0)^3}=2$

$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\dots = 0+1\cdot x+\frac{-1}{2!}x^2+\frac{2}{3!}x^2+\dots$

equal to

$x-x^2/2+x^3/3-x^4/4\dots$

$\ln(2)=\ln(1+1)=$

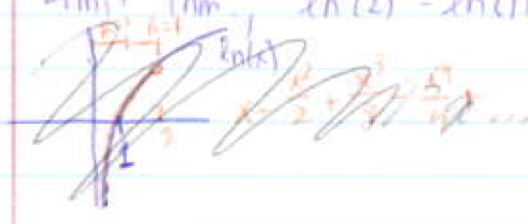
$\ln(1+x)=x-x^2/2+x^3/3-x^4/4\dots$ only guaranteed to converge for $|x|<R=1$

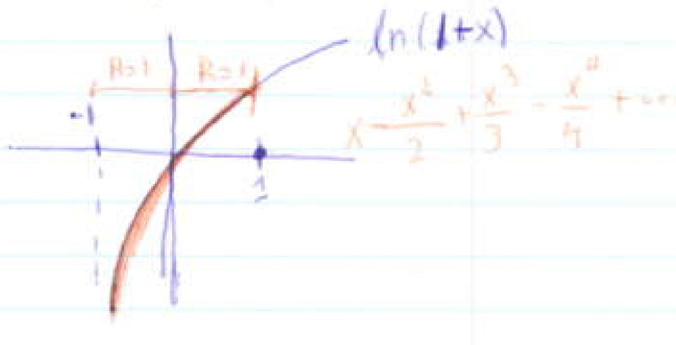
However $1-\frac{1^2}{2}+\frac{1^3}{3}-\frac{1^4}{4}+\dots=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ can be checked for converges: Use Alternating series test

$\frac{1}{n}\to 0$ as $n\to\infty$ &

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is $\searrow$

So $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\dots$ converges So by Abel's Limit Thm. $\ln(2)=\ln(1+1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$





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$$\tan^{-1}x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{1}{1-(-t^2)}\,dt = \int_0^x \overbrace{\left(1-t^2+t^4-t^6+t^8-\dots\right)}^{R=1}dt$$

$$\tan^{-1}x = \int_0^x 1\,dt - \int_0^x t^2\,dt + \int_0^x t^4\,dt - \dots = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\frac{\pi}{4} = \tan^{-1} = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \frac{1^9}{9} - \dots$$

$$\pi = 4 - 4/3 + 4/5 - 4/7 + 4/9 - \dots$$

Faster convergence: $\frac{\pi}{6} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{(1/\sqrt{3})^3}{3} + \frac{(1/\sqrt{3})^5}{5} - \dots$

$f(x) = (1+x)^{1/3}$	$f(0) = 1$
$f'(x) = \frac{1}{3}(1+x)^{1/3-1}$	$f'(0) = 1/3$
$f''(x) = \frac{1}{3}\left(\frac{1}{3}-1\right)(1+x)^{1/3-2}$	$f''(0) = \frac{1}{3}\left(\frac{1}{3}-1\right)$
$f'''(x) = \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)(1+x)^{1/3-3}$	$f'''(0) = \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)$
$f^{(4)}(x) = \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)(1+x)^{1/3-4}$	$f^{(4)}(0) = \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 1 + \frac{1/3}{1} + \frac{1/3(1/3-1)}{2!}x^2 + \frac{1/3(1/3-1)(1/3-2)}{3!}x^3 + \dots$$