Homogeneity and compactness: a mystery from set-theoretic topology

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Beyond metric spaces...

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Topological spaces are compact iff every net has a limit point iff every open cover has a finite subcover.

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Convention

All spaces are assumed to be Tychonoff (also known as $T_{3.5}$), i.e., assume that points and closed sets can be separated by continuous real-valued functions. If you don't know what this means, then forget I mentioned it.

Infinite products

Given a set A we write X^A for the set of all functions $f : A \to X$. We topologize X^A with the **product topology**, also known as the **topology of pointwise convergence**.

It works like this. A net $(f_i)_{i\in I}$ of functions in X^A converges to a function g in X^A iff the net $(f(a)_i)_{i\in I}$ converges to $g(a)$ for all $a \in A$.

Equivalently, the open sets of X^A are unions of sets of the form $\bigcap_{k=1}^{n} \pi_{a_k}^{-1} U_k$ where $n \in \mathbb{N}$, each U_k is open in X, each a_k is in A, and $\pi_{a_k}^{-1} U_k = \{f \in X^A : f(a_k) \in U_k\}$.

More generally, if we have a list $(X_a)_{a\in A}$ of spaces, we can form the product space $\prod_{a\in A} X_a$ whose elements are the maps f with domain A such that $f(a) \in X_a$ for all $a \in A$.

Compact products

Theorem (Tychonoff)

Products of compact spaces are compact.

Corollary

2^A *is always compact. (2 is the two-point space.)*

$2^{\mathbb{N}}$ is homeomorphic to the Cantor middle-thirds set.

Homogeneity

A map $f\colon X \to Y$ is **continuous** if f-preimages of open sets are open (or, equivalently, if f maps convergent nets to convergent nets).

A **homeomorphism** is a continuous bijection whose inverse is also continuous.

$\mathrm{Aut}(X)$ is the group of all **autohomeomorphisms** of X. (The group operation is composition.)

A space X is **homogeneous** if for all $p, q \in X$ there exists $h \in \mathrm{Aut}(X)$ such that $h(p) = q$. (In other words, $\mathrm{Aut}(X)$ acts transitively on X.) Informally, a homogeneous space looks the same no matter which point you stand on.

Examples

Every (discrete) finite space F is homogeneous because $\mathrm{Aut}(F)$ includes all permutations.

Every topological group G is homogeneous because $\mathrm{Aut}(G)$ includes all translations.

Products of homogeneous spaces are homogeneous. In particular, 2^A is always homogeneous and compact.

Cardinals

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Theorem (Schröder-Bernstein)

If $|A| \leq |B| \leq |A|$, then $|A| = |B|$.

Axiom of Choice

For all sets A and B, $|A| \leq |B|$ or $|B| \leq |A|$.

How high can you count?

We say A is **countable** if $|A| \leq |\mathbb{N}|$.

Theorem (Cantor)

$|A| < \left|2^A\right|$.

Corollary

$|2| < |\mathbb{Z}| = |\mathbb{N}| < \left|2^{\mathbb{N}}\right| = |\mathbb{R}| < \left|2^{\mathbb{R}}\right|$.

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Every set of cardinals has a minimum. (This follows from the Axiom of Foundation.)

Let $|A|^+$ denote the **successor cardinal** of A, which is the least cardinal greatest than $|A|$.

Cells

A **cellular family** of a space X is a set $\mathcal{A}$ of nonempty open subsets of X such that every two $U, V \in \mathcal{A}$ are disjoint.

The **cellularity** $c(X)$ of a space X is least cardinal κ such that every cellular family of X has size at most κ.

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- By an infinite version of the pigeonhole principle, uncountably many of the members of $\mathcal{G}$ have measure 2^{-n} for the same n.
- Thus, the union of some finite subfamily of $\mathcal{G}$ has measure greater than 1, which is absurd.

Isolation

Informally, the product topology of 2^A is very coarse. The open sets are just too big for us to lay out more than countably many of them side by side without overlap.

However, if we retopologize the set 2^A to be discrete (*i.e.*, all points are isolated) then we get a space D such that $c(D) = \left|2^A\right|$. This space is homogeneous, but not compact unless A is finite.

A more structured environment

If $f : X \to Y$ is a continuous onto map, then $c(X) \geq c(Y)$.

Theorem (Kuz'minov)

Every compact topological group G is a continuous image of 2^A for some A.

Corollary

$c(G) \leq |\mathbb{N}|$.

Every compact metric space M is a continuous image of $2^{\mathbb{N}}$, so $c(M) \leq |\mathbb{N}|$ too.

The Problem

Van Douwen's Problem, which is unsolved after over 30 years, asks if there is a compact homogeneous space X such that $c(X) > |\mathbb{R}|$.

In other words, every known compact homogeneous space X satisfies $c(X) \leq |\mathbb{R}|$, but we do not know if this is true in general.

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Theorem (Maurice)

There is a compact homogeneous space X satisfying $c(X) = |\mathbb{R}|$.

Maurice's spaces are higher order Cantor sets. One is $\left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ with the lexicographic order topology.

The brute-force approach...

It's not hard to compactify a given space X. Just take any space and "fill in the holes." Essentially, if a net doesn't have a limit point, then add one.

More precisely, we can always find a copy of X in a sufficiently large cube $[0, 1]^A$. We fill in the holes by taking the closure of this copy of X. This closure is compact because $[0, 1]^A$ is compact.

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It's even easier to "homogenize" a space. Given an infinite space X, use the following subspace of X^X.

$$\left\{ f \in X^X : \forall q \in X \quad |\{p \in X : f(p) = q\}| = |X| \right\}$$

...looks promising...

The above methods of compactifying and homogenizing never decrease cellularity.

Therefore, if we take a big discrete space like D satisfying $c(D) = \left|2^{\mathbb{R}}\right| > |\mathbb{R}|$, then we can get compact space C and a homogeneous space H such that $c(C) > |\mathbb{R}|$ and $c(H) > |\mathbb{R}|$.

. . . but doesn't work.

Unfortunately, compactifying by the above method generally breaks homogeneity and homogenizing by the above method always breaks compactness.

This is just the tip of the iceberg of why Van Douwen's Problem is hard.

Construction equipment

Given an arbitrary infinite list $(X_a)_{a \in A}$ of compact homogeneous spaces, how can we combine them to produce a single compact homogeneous space? We know only one way: taking products. (Actually, I found a second way; we'll come back to that.)

Unfortunately, given any product $\prod_{a \in A} X_a$ of compact spaces, we have $c\left(\prod_{a \in A} X_a\right) > |\mathbb{R}|$ iff $c\left(\prod_{a \in F} X_a\right) > |\mathbb{R}|$ for some finite $F \subseteq A$. (Why? It involves a technical lemma about uncountable families of finite sets...)

Thus, infinite products get us no further than finite products. Moreover, all finite products P of known examples of homogeneous compact spaces satisfy $c\,(P) \leq |\mathbb{R}|$.

A new tool

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Theorem (M.)

Given any compact homogeneous space X, there is quotient Q of $\mathbb{T} \times X^{\mathbb{T}}$ such that Q is compact, homogeneous, and path connected and $c(Q) \geq c(X)$.

Want the details?

- For each $p \in \mathbb{T}$, let S_p be the open semicircle with midpoint p.
- Given $(p, f), (q, g) \in \mathbb{T} \times X^{\mathbb{T}}$, declare $(p, f) \sim (q, g)$ if and only if $p = q$ and $f(r) = g(r)$ for all $r \in S_p$.
- Let Q be the set of $\sim$-equivalence classes of $\mathbb{T} \times X^{\mathbb{T}}$.
- Give Q the unique topology that makes $(p, f) \mapsto (p, f)/\sim$ a quotient map.

What's this good for?

If we could prove that there is no compact homogeneous *path connected* space X with $c(X) > |\mathbb{R}|$, then we could immediately conclude that Van Douwen's Problem has a negative solution.

For the right choice of X (specifically, any of Maurice's spaces), we can prove that Q is a new example of a compact homogeneous space, not constructable through any other known techniques.

Given an arbitrary list $(X_a)_{a \in A}$ of compact homogeneous spaces, the above technique also yields compact homogeneous path connected quotient spaces of $\mathbb{T} \times \left(\prod_{a \in A} X_A\right)^{\mathbb{T}}$ and similar spaces.

Quotients don't increase cellularity, so the above technique won't produce a positive solution to van Douwen's Problem.

Basic definitions

A **local base** at a point p in space X is a set $\mathcal{B}$ of open subsets of X such that $p \in B$ for all $B \in \mathcal{B}$, and for every open subset U of X, if $p \in U$, then there exists $B \in \mathcal{B}$ such that $B \subseteq U$.

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A π-**base** of a space X is a set $\mathcal{B}$ of nonempty open subsets of X such that for every nonempty open subset U of X, there exists $B \in \mathcal{B}$ such that $B \subseteq U$.

How many bosses do you have?

Definition (Peregudov)

Suppose $\mathcal{A}$ is a set of sets. Define the **Noetherian type** of $\mathcal{A}$ to be the least cardinal κ such that for all $U \in \mathcal{A}$, we have $|\{V \in \mathcal{A} : U \subseteq V\}| < \kappa$.

For example, given a descending sequence of sets $(U_n)_{n\in\mathbb{N}}$, the set $\{U_n : n \in \mathbb{N}\}$ has Noetherian type $|\mathbb{N}|$.

Given an ascending sequence of sets $(U_n)_{n\in\mathbb{N}}$, the set $\{U_n : n \in \mathbb{N}\}$ has Noetherian type $|\mathbb{N}|^+$, the least cardinal greater than $|\mathbb{N}|$.

Is that the best you can do?

Let the **Noetherian type** $\mathrm{Nt}(X)$ denote the least of the Noetherian types of the bases of X.

Let **Noetherian π-type** $\pi\mathrm{Nt}(X)$ denote the least of the Noetherian types of the π-bases of X.

Let **local Noetherian type** $\chi\mathrm{Nt}(p, X)$ denote the least of the Noetherian types of the local bases at p in X.

Let **local Noetherian type** $\chi\mathrm{Nt}(X)$ denote $\sup_{p\in X} \chi\mathrm{Nt}(p, X)$.

Correlation. . .

Every known homogeneous compact space X satisfies $c(X) \leq |\mathbb{R}|$.

Theorem (M.)

Every known compact homogeneous space X also satisfies $\mathrm{Nt}(X) \leq |\mathbb{R}|^+$, $\pi\mathrm{Nt}(X) \leq |\mathbb{N}|^+$, *and* $\chi\mathrm{Nt}(X) \leq |\mathbb{N}|$.

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If X is a compact group or compact metric space, then $c(X) \leq |\mathbb{N}|$.

Theorem (M.)

If X is a compact group or compact metric space, then $\mathrm{Nt}(X)$, $\pi\mathrm{Nt}(X)$, and $\chi\mathrm{Nt}(X)$ are all at most $|\mathbb{N}|$.

. . . and causation

The **Generalized Continuum Hypothesis** (GCH) says that $|2^A| = |A|^+$ for all infinite A. Gödel and Cohen proved that if the commonly accepted axioms of mathematics, ZFC, are consistent, then both $\mathrm{ZFC} + \mathrm{GCH}$ and $\mathrm{ZFC} + \neg\mathrm{GCH}$ are consistent.

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Theorem (M.)

Assuming GCH, every compact homogeneous space X satisfies $\chi\mathrm{Nt}\,(p, X) \leq c\,(X)$ for all $p \in X$.

Thus, assuming GCH, finding a compact homogeneous space X satisfying $\chi\mathrm{Nt}\,(p, X) > |\mathbb{R}|$ would positively solve Van Douwen's Problem.

Other directions

My published research also investigates:

- connectifying spaces,
- $\mathrm{Nt}(X)$ for various classes of (generally inhomogeneous) compacta,
- a more finely grained order-theoretic topological invariant, the Tukey class of a local base,
- $\mathrm{Nt}(\beta\mathbb{N}\setminus\mathbb{N})$, $\pi\mathrm{Nt}(\beta\mathbb{N}\setminus\mathbb{N})$, $\chi\mathrm{Nt}(\beta\mathbb{N}\setminus\mathbb{N})$, and
- Tukey classes of local bases in $\beta\mathbb{N}\setminus\mathbb{N}$.

The deciders

Definition

A set $\mathcal{U}$ of subsets of $\mathbb{N}$ is a nonprincipal **ultrafilter** on $\mathbb{N}$ if:

- Every $A \in \mathcal{U}$ is infinite.
- For all $A, B \in \mathcal{U}$, we have $A \cap B \in \mathcal{U}$.
- For all $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

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For every ultrafilter $\mathcal{U}$ and sequence $(x_n)_{n\in\mathbb{N}}$ on $[0,1]$, there is a unique $L \in [0,1]$ such that for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| < \varepsilon\}$ is in $\mathcal{U}$. We call L the $\mathcal{U}$-limit of $(x_n)_{n\in\mathbb{N}}$ and write $L = \lim_{n\to\mathcal{U}} x_n$.

(More generally, $\mathcal{U}$-limits always exist in compact spaces.)

It's all about the combinatorics

- Topologize the space of nonprincipal ultrafilters on $\mathbb{N}$ by declaring every open set to be a union of sets of the form $\{\mathcal{U} : A \in \mathcal{U}\}$ where $A \subseteq \mathbb{N}$.
- Equivalently, a net $(\mathcal{V}_i)_{i\in I}$ converges to $\mathcal{U}$ if and only if for every $A \in \mathcal{U}$, there is some $i \in I$ such that $A \in \mathcal{V}_j$ for all $j \geq i$.
- This space of ultrafilters is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$, so, for the purposes of topology, it *is* $\beta\mathbb{N} \setminus \mathbb{N}$.

Questions about $\beta\mathbb{N} \setminus \mathbb{N}$ boil down to combinatorial questions about sets of subsets of $\mathbb{N}$.

What is $\beta\mathbb{N} \setminus \mathbb{N}$ like?

- $\beta\mathbb{N} \setminus \mathbb{N}$ is compact.
- (Frolik) $\beta\mathbb{N} \setminus \mathbb{N}$ is not homogeneous.
- $|\beta\mathbb{N} \setminus \mathbb{N}| = \left|2^{\mathbb{R}}\right|$.
- $c\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = |\mathbb{R}|$.
- $\beta\mathbb{N} \setminus \mathbb{N}$ has convergent nets, but no nontrivial convergent sequences. (In particular, $\beta\mathbb{N} \setminus \mathbb{N}$ is not metrizable.)

Increasing precision

Theorem (Malykhin)

- $|\mathbb{N}|^+ \leq \pi\mathrm{Nt}\,(\beta\mathbb{N} \setminus \mathbb{N}) \leq \mathrm{Nt}\,(\beta\mathbb{N} \setminus \mathbb{N})$.
- $|\mathbb{R}| = |\mathbb{N}|^+ \Rightarrow \pi\mathrm{Nt}\,(\beta\mathbb{N} \setminus \mathbb{N}) = \mathrm{Nt}\,(\beta\mathbb{N} \setminus \mathbb{N}) = |\mathbb{R}|$.

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- $|\mathbb{R}| = |\mathbb{N}|^+ \Rightarrow \pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = \mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = |\mathbb{R}|$.

Theorem (M.)

- $\pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) \leq |\mathbb{R}|$ and $\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) \leq |\mathbb{R}|^+$.
- $\chi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) \leq \min\left\{\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right), |\mathbb{R}|\right\}$.
- $\pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = |\mathbb{R}| \Rightarrow \mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = |\mathbb{R}|$.

Moreover, ZFC is consistent with each of the following.

- $|\mathbb{N}|^+ = \pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = \chi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = \mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) < |\mathbb{R}|$.
- $|\mathbb{N}|^+ < \pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = \chi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = \mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) < |\mathbb{R}|$.
- $|\mathbb{N}|^+ = \pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) < \mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) < |\mathbb{R}|$.
- $|\mathbb{N}|^+ < \pi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) < \chi\mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right) = |\mathbb{R}| < \mathrm{Nt}\left(\beta\mathbb{N} \setminus \mathbb{N}\right)$.

What is $\beta\mathbb{N} \setminus \mathbb{N}$, really?

Defintion

A compact space Y is a **compactification** of a space X if a dense subspace of Y is homeomorphic to X.

- Every space X has a unique compactification βX that is maximal in the sense that for every compactification γX of X, there is a continuous surjection $f\colon \beta X \to \gamma X$ such that f restricted to X is the identity map.
- To build βX, let $\mathcal{F}$ be the set of all continuous maps from X to $[0,1]$. This makes X homeomorphic to the subspace $\tilde{X} = \{(f(p))_{f\in\mathcal{F}} : p \in X\}$ of $[0,1]^{\mathcal{F}}$. The closure of $\tilde{X}$ in $[0,1]^{\mathcal{F}}$ is βX.
- We abuse notation and equate $\mathbb{N}$ and $\tilde{\mathbb{N}}$. Thus, $\beta\mathbb{N} \setminus \mathbb{N}$ is just $\beta\mathbb{N} \setminus \tilde{\mathbb{N}}$.

What happened to the ultrafilters?

There is a natural homeomorphism h from the space of nonprincipal ultrafilters on $\mathbb{N}$ to the space $\beta\mathbb{N} \setminus \mathbb{N}$.

- For each ultrafilter $\mathcal{U}$, define $h(\mathcal{U})$ by $h(\mathcal{U})(f) = \lim_{n \to \mathcal{U}} f(n)$ for all $f : \mathbb{N} \to [0, 1]$.

There is a natural description of h^{-1}.

- For each $x \in \beta\mathbb{N} \setminus \mathbb{N}$, $h^{-1}(x)$ is the set of all sets of the form $\{n : (f(n))_{f \in \mathcal{F}} \in V\}$ where V is a neighborhood of x.