

# GO-SPACES AND NOETHERIAN SPECTRA

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ABSTRACT. The Noetherian type of a space is the least  $\kappa$  for which the space has a  $\kappa^{\text{op}}$ -like base, *i.e.*, a base in which no element has  $\kappa$ -many supersets. We prove some results about Noetherian types of (generalized) ordered spaces and products thereof. For example: the density of a product of not-too-many compact linear orders never exceeds its Noetherian type, with equality possible only for singular Noetherian types; we prove a similar result for products of Lindelöf GO-spaces. A countable product of compact linear orders has an  $\omega_1^{\text{op}}$ -like base if and only if it is metrizable. (It is known that every metrizable space has an  $\omega^{\text{op}}$ -like base.) An infinite cardinal  $\kappa$  is the Noetherian type of a compact LOTS if and only if  $\kappa \neq \omega_1$  and  $\kappa$  is not weakly inaccessible. There is a Lindelöf LOTS with Noetherian type  $\omega_1$  and there consistently is a Lindelöf LOTS with weakly inaccessible Noetherian type.

## 1. INTRODUCTION

The Noetherian type of a topological space is an order-theoretic analog of its weight.

**Definition 1.1.** Given a cardinal  $\kappa$ , define a poset to be  $\kappa^{\text{op}}$ -like if no element is below  $\kappa$ -many elements.

In the context of families of subsets of a topological space, we will always implicitly order by inclusion. For example, a descending chain of open sets of type  $\omega$  is  $\omega^{\text{op}}$ -like; an ascending chain of open sets of type  $\omega$  is  $\omega_1^{\text{op}}$ -like but not  $\omega^{\text{op}}$ -like.

**Definition 1.2.** Given a space  $X$ ,

- the *weight* of  $X$ , or  $w(X)$ , is the least  $\kappa \geq \omega$  such that  $X$  has a base of size at most  $\kappa$ ;
- the *Noetherian type* of  $X$ , or  $Nt(X)$ , is the least  $\kappa \geq \omega$  such that  $X$  has a base that is  $\kappa^{\text{op}}$ -like.

Equivalently,  $Nt(X)$  is the least  $\kappa \geq \omega$  such that  $X$  has a base  $\mathcal{B}$  such that  $\bigcap \mathcal{A}$  has empty interior for all  $\mathcal{A} \in [\mathcal{B}]^\kappa$ .

In 1997, Noetherian type was introduced by Peregudov [12]. Preceding this introduction are several papers by Peregudov, Šapirovskii and Malykhin [7, 10, 11, 13] about the topological properties  $Nt(\cdot) = \omega$  and  $Nt(\cdot) \leq \omega_1$ . In 1998 Bennett and Lutzer [3] rediscovered (and renamed) the property  $Nt(\cdot) = \omega$  and proved (among other things) that a GO-space  $X$  is metrizable if and only if  $Nt(X) = \omega$ . Balogh, Bennett, Burke, Gruenhagen, Lutzer, and Mashburn [2], and Bailey [1] further investigated the property  $Nt(\cdot) = \omega$  in the context of related base properties.

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More recently, the author has extensively investigated the Noetherian type of  $\beta\mathbb{N} \setminus \mathbb{N}$  [9] and the Noetherian types of homogeneous compacta and dyadic compacta [8]. (See Engelking [4], Juhász [5], and Kunen [6] for all undefined terms.)

A surprising result from [8] is that no dyadic compactum has Noetherian type  $\omega_1$ . In other words, given an  $\omega_1^{\text{op}}$ -like base of a dyadic compactum  $X$ , one can construct an  $\omega^{\text{op}}$ -like base of  $X$ . This result does not generalize to all compacta. In that same paper, it was shown how to construct a compactum with Noetherian type  $\kappa$ , for any infinite cardinal  $\kappa$ . It is still an open problem whether any infinite cardinals other than  $\omega_1$  are excluded from the spectrum of Noetherian types of dyadic compacta, although it was shown that the Noetherian types of dyadic compacta include  $\omega$ , all singular cardinals, and  $\kappa^+$  for every infinite cardinal  $\kappa$  with uncountable cofinality.

*Question 1.3.* If  $\kappa$  is a singular cardinal with cofinality  $\omega$ , then is there a dyadic compactum with Noetherian type  $\kappa^+$ ? Is there a dyadic compactum with weakly inaccessible Noetherian type?

The above two questions are typical of the “sup=max” problems of set-theoretic topology. See Juhász [5] for a systematic study of these problems.

Though the above two questions remain open problems, we can now answer the corresponding questions for compact linear orders. The spectrum of Noetherian types of linearly ordered compacta includes  $\omega$ , excludes  $\omega_1$ , includes all singular cardinals, includes  $\kappa^+$  for all uncountable cardinals  $\kappa$ , and excludes all weakly inaccessible cardinals. In the process of proving this claim, we will prove a general technical lemma which says roughly that if  $X$  is a product of not-too-many  $\mu$ -compact GO-spaces for some fixed cardinal  $\mu$ , then  $d(X) \leq Nt(X)$  and in most cases  $d(X) < Nt(X)$ .

**Definition 1.4.**

- A space  $X$  is  $\kappa$ -compact if  $\kappa$  is a cardinal and every open cover of  $X$  has a subcover of size less than  $\kappa$ .
- A *GO-space*, or generalized ordered space, is a subspace of a linearly ordered topological space. Equivalently, a GO-space is a linear order with a topology that has a base consisting only of convex sets.
- The *density*  $d(X)$  of a space  $X$  is the least infinite cardinal  $\kappa$  such that  $X$  has a dense subset of size at most  $\kappa$ .

It is natural to ask what happens to the spectrum of Noetherian types of compact linear orders if we gently relax the assumption of compactness. It turns out that there are Lindelöf linear orders with Noetherian type  $\omega_1$ , and, less expectedly, that it is consistent (relative to existence of an inaccessible cardinal) that there is a Lindelöf linear order with weakly inaccessible Noetherian type. However, it is not consistent for a Lindelöf GO-space to have strongly inaccessible Noetherian type.

We also consider the relationship between metrizable and Noetherian type, focusing on GO-spaces. Bennett and Lutzer [3] noted that every metrizable space has Noetherian type  $\omega$ , and proved the (more difficult) converse for GO-spaces. We strengthen these results as follows. For a Lindelöf GO-space  $X$ , we show that  $X$  is metrizable if and only if  $Nt(X) = \omega$  if and only if  $Nt(X) = \omega_1$  and  $X$  is separable. For a countable product  $X$  of compact linear orders,  $X$  is metrizable if and only if  $Nt(X) = \omega$  if and only if  $Nt(X) = \omega_1$ . (Note that every Lindelöf metric space is separable, and every compact GO-space is a compact linear order.)

## 2. SMALL DENSITIES AND LARGE NOETHERIAN TYPES

**Definition 2.1.** The  $\pi$ -weight  $\pi(X)$  of a space  $X$  is the least infinite cardinal  $\kappa$  such that a space has  $\pi$ -base of size at most  $\kappa$ ;

**Proposition 2.2.** [12] *If  $X$  is a space and  $\pi(X) < \text{cf } \kappa \leq \kappa \leq w(X)$ , then  $Nt(X) > \kappa$ .*

*Proof.* Suppose  $\mathcal{A}$  is a base of  $X$  and  $\mathcal{B}$  is  $\pi$ -base of  $X$  of size at most  $\pi(X)$ . We then have  $|\mathcal{A}| \geq \kappa$ ; hence, there exist  $\mathcal{U} \in [\mathcal{A}]^\kappa$  and  $V \in \mathcal{B}$  such that  $V \subseteq \bigcap \mathcal{U}$ . Hence, there exists  $W \in \mathcal{A}$  such that  $W \subseteq V \subseteq \bigcap \mathcal{U}$ ; hence,  $\mathcal{A}$  is not  $\kappa^{\text{op}}$ -like.  $\square$

Note that if  $X$  is a product of at most  $d(X)$ -many GO-spaces, then  $\pi(X) = d(X)$  is witnessed by the following construction. For any  $D$  (topologically) dense in  $X$  and of minimal size, collect all the finitely supported products of topologically open intervals with endpoints from the union of  $\{\pm\infty\}$  and the set of all coordinates of points from  $D$ .

Trivially,  $Nt(X) \leq w(X)^+$  for all spaces  $X$ . The next example shows that this upper bound is attained.

**Example 2.3.** [12] The double-arrow space, defined as  $((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\})$  ordered lexicographically, has  $\pi$ -weight  $\omega$  and weight  $2^{\aleph_0}$ . By Proposition 2.2, it has Noetherian type  $(2^{\aleph_0})^+$ .

## 3. LINDELÖF GO-SPACES

Bennett and Lutzer [3] noted that “it is easy to prove that any metric space, and indeed any metacompact Moore space has an OIF base,” *i.e.*, has Noetherian type  $\omega$ . The following proof is for the reader’s convenience.

**Theorem 3.1.** *Every metric space has an  $\omega^{\text{op}}$ -like base.*

*Proof.* Let  $X$  be a metric space. For each  $n < \omega$ , let  $\mathcal{A}_n$  be a locally finite open refinement of the (open) balls of radius  $2^{-n}$  in  $X$ . Set  $\mathcal{A} = \bigcup_{n < \omega} \mathcal{A}_n \setminus \{\emptyset\}$ . The set  $\mathcal{A}$  is a base of  $X$  because if  $p \in X$  and  $n < \omega$ , then there exists  $U \in \mathcal{A}_{n+1}$  such that  $p \in U$  and  $U$  is contained in the ball of radius  $2^{-n}$  with center  $p$ . Let us show that  $\mathcal{A}$  is  $\omega^{\text{op}}$ -like. Suppose that  $m < \omega$ ,  $U \in \mathcal{A}$ ,  $V \in \mathcal{A}_m$ , and  $U \subsetneq V$ . There then exist  $p \in U$  and  $\epsilon_0 > \epsilon_1 > 0$  such that the  $\epsilon_0$ -ball with center  $p$  is contained in  $U$  and the  $\epsilon_1$ -ball with center  $p$  intersects only finitely many elements of  $\mathcal{A}_n$  for all  $n < \omega$  satisfying  $2^{-n} > \epsilon_0/2$ . If  $2^{-m} \leq \epsilon_0/2$ , then  $V$  is contained in the  $\epsilon_0$ -ball with center  $p$ , in contradiction with  $U \subsetneq V$ . Hence,  $2^{-m} > \epsilon_0/2$ ; hence, there are only finitely many possibilities for  $m$  and  $V$  given  $U$ , for  $V$  intersects the  $\epsilon_1$ -ball with center  $p$ .  $\square$

Conversely, Bennett and Lutzer [3] proved that GO-spaces with Noetherian type  $\omega$  are metrizable. Theorem 3.5 proves a slightly stronger statement for Lindelöf GO-spaces.

**Lemma 3.2.** *Let  $X$  be a Lindelöf GO-space with open cover  $\mathcal{A}$ . The cover  $\mathcal{A}$  has a countable, locally finite refinement consisting only of open convex sets.*

*Proof.* Let  $\{A_n : n < \omega\}$  be a countable refinement of  $\mathcal{A}$  consisting only of open convex sets. For each  $n < \omega$ , set  $B_n = A_n \setminus \bigcup_{m < n} A_m$ ; set  $\mathcal{B} = \{B_n : n < \omega\}$ . The set  $\mathcal{B}$  is a locally finite refinement of  $\mathcal{A}$ . Let  $\mathcal{C}$  be the set of open convex subsets

of  $X$  which intersect only finitely many elements of  $\mathcal{B}$ . Let  $\mathcal{D}$  be the set of  $U \in \mathcal{C}$  satisfying  $\overline{U} \subseteq V$  for some  $V \in \mathcal{C}$ . Let  $\{D_n : n < \omega\}$  be a countable subcover of  $\mathcal{D}$ . For each  $n < \omega$ , set  $E_n = D_n \setminus \bigcup_{m < n} D_m$ ; set  $\mathcal{E} = \{E_n : n < \omega\}$ . The set  $\mathcal{E}$  is a locally finite refinement of  $\mathcal{C}$ . For each  $n < \omega$ , set  $F_n = A_n \setminus \bigcup\{E \in \mathcal{E} : B_n \cap E = \emptyset\}$ ; set  $\mathcal{F} = \{F_n : n < \omega\}$ . Since  $\mathcal{E}$  is locally finite, each  $F_n$  is open. Moreover,  $F_n \subseteq A_n$  for all  $n < \omega$ ; hence,  $\mathcal{F}$  is a refinement of  $\mathcal{A}$ .

Let  $\mathcal{G} = \bigcup_{n < \omega} \mathcal{G}_n$  where  $\mathcal{G}_n$  is the set of maximal convex subsets of  $F_n$ . This makes  $\mathcal{G}$  an open refinement of  $\mathcal{A}$ . It suffices to show that  $\mathcal{G}$  is locally finite and countable. We accomplish this in three steps. First, we show that  $\mathcal{F}$  is locally finite. Second, we show that each  $\mathcal{G}_n$  is locally finite. Third, we show that each  $\mathcal{G}_n$  is countable. This suffices because each  $\mathcal{G}_n$  is a partition of  $F_n$ .

For the first step, since  $\mathcal{E}$  is a locally finite cover of  $X$ , it suffices to show that each element of  $\mathcal{E}$  only intersects finitely many elements of  $\mathcal{F}$ . Let  $i < \omega$  and choose  $V \in \mathcal{C}$  such that  $E_i \subseteq V$ . Suppose  $j < \omega$  and  $E_i \cap F_j \neq \emptyset$ . We then have  $E_i \cap B_j \neq \emptyset$  by definition of  $F_j$ . Hence,  $V \cap B_j \neq \emptyset$ ; hence, there are only finitely many possibilities for  $B_j$ ; hence, there are only finitely many possibilities for  $F_j$ .

For the second step, suppose  $p \in X$ . We just need to show that  $p$  has a neighborhood intersecting at most finitely many  $G \in \mathcal{G}_n$ . Let  $U$  be an open convex neighborhood of  $p$  intersecting at most finitely many elements of  $\mathcal{E}$ . By its construction, each  $E_i \in \mathcal{E}$  is a finite union of convex sets. Moreover, by the construction of  $F_n$ , given any  $G_0, G_1 \in \mathcal{G}_n$  with  $G_0 < G_1$ , there exists a maximal convex subset  $H$  of some  $E_i \in \mathcal{E}$  such that  $G_0 < H < G_1$ . Therefore,  $U$  intersects at most finitely many elements of  $\mathcal{G}_n$ .

For the third step, we will extend  $\mathcal{G}_n$  to an open cover  $\mathcal{U}$  of  $X$  such that every subcover of  $\mathcal{U}$  includes  $\mathcal{G}_n$ . This suffices because  $X$  is Lindelöf. For each  $G \in \mathcal{G}_n$ , choose  $p_G \in G$ . For each  $q \in F_n$ , let  $U(q) = G$  where  $q \in G \in \mathcal{G}_n$ . For each  $q \in X \setminus F_n$ , use the local finiteness of  $\mathcal{G}_n$  to find an open neighborhood  $U(q)$  of  $q$  such that  $p_G \notin U(q)$  for all  $G \in \mathcal{G}_n$ . Set  $\mathcal{U} = \{U(q) : q \in X\}$ , which is our desired open cover.  $\square$

**Definition 3.3.** Let  $H_\theta$  denote the set of all sets that are hereditarily of size less  $\theta$ , where  $\theta$  is a regular cardinal sufficiently large for the argument at hand. The relation  $M \prec H_\theta$  means that  $\langle M, \in \rangle$  is an elementary substructure of  $\langle H_\theta, \in \rangle$ .

**Lemma 3.4.** *Let  $X$  be a nonseparable, Lindelöf GO-space. The space  $X$  does not have an  $\omega^{\text{op}}$ -like base.*

*Proof.* Lindelöf metric spaces are separable, so  $X$  is not metrizable, so by Bennett and Lutzer's result,  $Nt(X) > \omega$ . We also provide an independently discovered proof below. This proof is direct in that the property of metrizability is not used in the argument.

Let  $\mathcal{A}$  be a base of  $X$ . Let us show that  $\mathcal{A}$  is not  $\omega^{\text{op}}$ -like. First, let us construct sequences of open sets  $\langle A_{n,k} \rangle_{n,k < \omega}$  and  $\langle B_{n,k} \rangle_{n,k < \omega}$ . Our requirements are that  $B_{n,i} \subseteq A_{n,i} \in \mathcal{A}$ , that  $B_{n,i}$  is convex, that  $\{B_{n,k} : k < \omega\}$  is a locally finite cover of  $X$  and pairwise  $\subseteq$ -incomparable, and that  $\{A_{i,k} : k < \omega\} \cap \{A_{j,k} : k < \omega\} \subseteq [X]^1$  for all  $i < j < \omega$  and  $n < \omega$ .

Suppose  $n < \omega$  and we are given  $\langle A_{m,k} \rangle_{k < \omega}$  and  $\langle B_{m,k} \rangle_{k < \omega}$  for all  $m < n$  and they meet our requirements. Let  $p \in X$ . Set  $V_p = \bigcap \{B_{m,k} : m < n \text{ and } k < \omega \text{ and } p \in B_{m,k}\}$ . The set  $V_p$  is open. If  $|V_p| = 1$ , then set  $U_p = V_p$ . If  $|V_p| > 1$ , then choose  $U_p \in \mathcal{A}$  such that  $p \in U_p \subsetneq V_p$ . Set  $\mathcal{U} = \{U_p : p \in X\}$ . By Lemma 3.2,

there exists a countable, locally finite refinement  $\mathcal{B}_n$  of  $\mathcal{U}$  consisting only of convex sets. Since  $\mathcal{B}_n$  is locally finite, it has no infinite ascending chains; hence, we may assume  $\mathcal{B}_n$  is pairwise  $\subseteq$ -incomparable because we may shrink  $\mathcal{B}_n$  to its maximal elements. Let  $\{B_{n,k} : k < \omega\} = \mathcal{B}_n$ . For each  $k < \omega$ , set  $A_{n,k} = U_p$  for some  $p \in X$  satisfying  $B_{n,k} \subseteq U_p$ . Suppose  $m < n$  and  $i, j < \omega$  and  $A_{m,i} = A_{n,j} \notin [X]^1$ . Choose  $p \in X$  such that  $A_{n,j} = U_p$ ; choose  $k < \omega$  such that  $p \in B_{m,k}$ . We then have  $B_{m,i} \subseteq A_{m,i} = U_p \subsetneq V_p \subseteq B_{m,k}$ , in contradiction with the pairwise  $\subseteq$ -incomparability of  $\{B_{m,l} : l < \omega\}$ . Thus,  $\{A_{m,l} : l < \omega\} \cap \{A_{n,l} : l < \omega\} \subseteq [X]^1$  for all  $m < n$ . By induction,  $\langle A_{n,k} \rangle_{n,k < \omega}$  and  $\langle B_{n,k} \rangle_{n,k < \omega}$  meet our requirements.

Let  $\{X, \leq, \mathcal{A}, \vec{A}, \vec{B}\} \subseteq M \prec H_\theta$  and  $|M| = \omega$ . Since  $X$  is nonseparable, there must be a nonempty open convex set  $W$  disjoint from  $M$ . For each  $n < \omega$ , choose  $i_n < \omega$  such that  $W \cap B_{n,i_n} \neq \emptyset$ . Let us show that  $W \subseteq A_{n,i_n} \notin [X]^1$  for all  $n < \omega$ . Fix  $n < \omega$ . If  $A_{n,i_n} \in [X]^1$ , then  $A_{n,i_n} \subseteq M$ , which implies  $W \cap M \neq \emptyset$ , which is absurd. Therefore, let us show that  $W \subseteq A_{n,i_n}$ . Seeking a contradiction, suppose  $W \not\subseteq A_{n,i_n}$ . We then have  $W \not\subseteq B_{n,i_n}$ . Since  $X$  is Lindelöf,  $\{p \in X : p < B_{n,i_n}\}$  has a countable cofinal subset  $Y$ . By elementarity, we may choose  $Y \in M$ . Since  $Y$  is countable,  $Y \subseteq M$ . Likewise,  $\{p \in X : B_{n,i_n} < p\}$  has a coinital subset  $Z$  with  $Z \subseteq M$ . Since  $W$  and  $B_{n,i_n}$  are convex and  $W$  intersects both  $B_{n,i_n}$  and  $X \setminus B_{n,i_n}$ ,  $W$  intersects  $Y$  or intersects  $Z$ . Hence,  $W$  intersects  $M$ , which yields our desired contradiction.

We have shown that  $W \subseteq A_{n,i_n}$  for all  $n$ , and hence that  $\mathcal{A}$  is not  $\omega^{\text{op}}$ -like.  $\square$

**Theorem 3.5.** *Let  $X$  be a Lindelöf GO-space. The following are equivalent.*

- (1)  $X$  is metrizable.
- (2)  $X$  has an  $\omega^{\text{op}}$ -like base.
- (3)  $X$  is separable and has an  $\omega_1^{\text{op}}$ -like base.

*Proof.* By Theorem 3.1, (1) implies (2). By Lemma 3.4, (2) implies (3). Hence, it suffices to show that (3) implies (1). Suppose  $X$  has a countable dense subset  $D$  and an  $\omega_1^{\text{op}}$ -like base. We then have  $\pi(X) = \omega$ ; hence, by Proposition 2.2,  $w(X) = \omega$ ; hence,  $X$  is metrizable.  $\square$

See Example 6.1 for a nonseparable Lindelöf linear order that has Noetherian type  $\omega_1$ .

#### 4. SMALL NOETHERIAN TYPES AND SMALLER DENSITIES

For compact linearly ordered topological spaces, the theorem at the end of this section strengthens Theorem 3.5. To prepare for this theorem, we first prove our main technical lemma, which we state in very general terms.

**Lemma 4.1.** *Suppose  $\kappa$  is a regular uncountable cardinal,  $\mu$  is an infinite cardinal,  $|\lambda^{<\mu}| < \kappa$  for all  $\lambda < \kappa$ ,  $X$  is a product of fewer than  $\kappa$ -many  $\mu$ -compact GO-spaces, and  $Nt(X) \leq \kappa$ . We then have  $d(X) < \kappa$ .*

*Proof.* Let  $X = \prod_{i < \nu} X_i$  where  $\nu < \kappa$  and each  $X_i$  is a  $\mu$ -compact subspace of a linearly ordered topological space  $Y_i$ . Seeking a contradiction, suppose  $d(X) \geq \kappa$ . Let  $\mathcal{U}$  be a  $\kappa^{\text{op}}$ -like base of  $X$ ,  $\{\langle Y_i, \leq_{Y_i}, X_i : i < \nu \rangle, \mathcal{U}\} \in M \prec H_\theta$ ,  $|M| < \kappa$ ,  $M \cap \kappa \in \kappa$ , and  $M^{<\mu} \subseteq M$ . (We can construct  $M$  as the union of an appropriate elementary chain of length  $\rho$ , where  $\rho$  is the least regular cardinal  $\geq \mu$ . Such an  $M$  is not too large because  $\rho < \kappa$ , a fact that follows from  $\mu \leq |2^{<\mu}| < \kappa$  and  $\text{cf}(\mu) <$

$\mu \Rightarrow \mu^+ \leq |\mu^{\text{cf}(\mu)}| < \kappa$ ). Since  $d(X) > |M|$ , there is a finite subproduct  $\prod_{i \in \sigma} X_i$  of  $X$  that has a nonempty open subset disjoint from  $M$ . We may choose this open subset to be the interior of a set of the form  $B = \prod_{i \in \sigma} B_i$  where each  $B_i$  is maximal among the convex subsets of  $X_i$  disjoint from  $M$ . Set  $A_i = \{p \in X_i : p < B_i\}$  and  $C_i = \{p \in X_i : p > B_i\}$ . Since  $\{p : A_i < p < C_i\} = B_i$ , which is nonempty but disjoint from  $M$ , we have  $\{A_i, C_i\} \not\subseteq M$  by elementarity.

**Claim.**  $\max\{\text{cf}(A_i), \text{ci}(C_i)\} \geq \mu$  for all  $i \in \sigma$ .

*Proof.* Seeking a contradiction, suppose  $\text{cf}(A_i) < \mu$  and  $\text{ci}(C_i) < \mu$ . We then have  $A_i \cap M, C_i \cap M \in M$ . Since  $A_i \cap M$  is cofinal in  $A_i$ ,  $A_i = \{p : \exists q \in A_i \cap M \ p \leq q\} \in M$ . Likewise,  $C_i \in M$ , in contradiction with the fact that  $\{A_i, C_i\} \not\subseteq M$ .  $\square$

Therefore, we may assume that  $\text{cf}(A_i) \geq \mu$  for all  $i \in \sigma$  (by symmetry). Since  $X$  is  $\mu$ -compact, there exists  $x_i = \sup_{Y_i}(A_i) = \min(B_i) \in X_i$  for all  $i \in \sigma$ .

**Claim.** *There exists  $y_i = \sup_{Y_i}(B_i) \in (x_i, \infty)$  for all  $i \in \sigma$ , with the understanding that in this proof all intervals are intervals of  $X_i$  (so  $y_i \in X_i$ ).*

*Proof.* If  $\text{ci}(C_i) \geq \mu$ , then, by  $\mu$ -compactness, there exists  $y_i = \inf_{Y_i}(C_i) \in X_i$ . In this case,  $y_i$  is also  $\max(B_i)$  because  $y_i \notin C_i$ . Moreover,  $y_i = \max(B_i) > \min(B_i) = x_i$  because otherwise the interior of  $B$  would be empty, for  $x_i = \sup_{Y_i}(A_i)$ , which is not an isolated point in  $X_i$ . If  $\text{ci}(C_i) < \mu$ , then  $C_i \in M$ , just as in the previous claim's proof, so there exists  $D_i \in M$  such that  $D_i$  is a cofinal subset of  $\{p \in X_i : p < C_i\}$  of minimal size. In this case,  $D_i$  includes a cofinal subset of  $B_i$ , so  $D_i \not\subseteq M$ , so  $|D_i| \geq \kappa$ , so  $\mu < \kappa \leq |D_i| = \text{cf}(B_i)$ , so there exists  $y_i = \sup_{Y_i}(B_i) \in X_i$  by  $\mu$ -compactness. Also,  $\text{cf}(B_i) \geq \kappa$  implies  $\sup_{Y_i}(B_i) > \min(B_i) = x_i$ . Thus, in any case there exists  $y_i = \sup(B_i) \in (x_i, \infty)$  for all  $i \in \sigma$ .  $\square$

Let  $U \in \mathcal{U}$  satisfy  $x_i \in \pi_i[U] \subseteq (-\infty, y_i)$  for all  $i \in \sigma$ . Since  $\text{cf}(A_i) \geq \mu > 1$  for all  $i \in \sigma$ , we then have  $\langle x_i : i \in \sigma \rangle \in \prod_{i \in \sigma} W_i \subseteq \pi_\sigma[U]$  where each  $W_i$  is of the form  $(u_i, v_i)$  or  $[u_i, v_i]$  for some  $u_i < x_i$  and  $v_i \leq y_i$ ; we may assume  $u_i \in M$ . Moreover, there exist  $p_i, q_i \in X_i \cap M$  such that  $u_i < p_i < q_i < x_i$ . Since  $\mathcal{U}$  is a  $\kappa^{\text{op}}$ -like base, it includes fewer than  $\kappa$ -many supersets of  $\bigcap_{i \in \sigma} \pi_i^{-1}[(u_i, q_i)]$  as members. Since the set of supersets of  $\bigcap_{i \in \sigma} \pi_i^{-1}[(u_i, q_i)]$  in  $\mathcal{U}$  is a set in  $M$  and a set of size less than  $\kappa$ , it is also a subset of  $M$ . In particular,  $U \in M$ .

Fix an arbitrary  $i \in \sigma$ . If  $\text{cf}(\pi_i[U]) < \mu$ , then  $M$  would include a cofinal subset of  $\pi_i[U]$ , in contradiction with  $B_i$  missing  $M$ . Therefore,  $\text{cf}(\pi_i[U]) \geq \mu$ . Hence, there exists  $z = \sup_{Y_i}(\pi_i[U])$ . By elementarity,  $z \in M$ , so  $B_i < z$ , so  $z = \min(C_i) = \sup_{Y_i}(B_i) = y$ . Because of the freedom in how we chose  $U$ , it follows that every neighborhood of  $x_i$  includes a neighborhood that, like  $\pi_i[U]$ , has supremum  $y_i$  (in  $Y_i$ ) and has cofinality at least  $\mu$ . Therefore, there is an infinite increasing sequence of points between  $x_i$  and  $y_i$  that are contained in every neighborhood of  $x_i$ , in contradiction with  $X_i$  being a subspace of the ordered space  $Y_i$ . Thus,  $d(X) < \kappa$ .  $\square$

**Corollary 4.2.** *Suppose that  $X$  is a product of at most  $2^{\aleph_0}$ -many Lindelöf GO-spaces such that  $\text{Nt}(X) \leq (2^{\aleph_0})^+$ . We then have  $d(X) \leq 2^{\aleph_0}$ .*

**Corollary 4.3.** *Suppose that  $\kappa$  is a regular uncountable cardinal,  $X$  is a product of less than  $\kappa$ -many linearly ordered compacta, and  $\text{Nt}(X) \leq \kappa$ . We then have  $d(X) < \kappa$ .*

The last corollary fails for singular  $\kappa$ . As we shall see in Theorem 5.1, if  $\lambda$  is an uncountable singular cardinal, then  $Nt(\lambda+1) = \lambda$ , despite the fact that  $d(\kappa+1) = \kappa$  for all infinite cardinals  $\kappa$ . Moreover, the space  $(\kappa+1)^\kappa$  has Noetherian type  $\omega$  and density  $\kappa$  for all infinite cardinals  $\kappa$ , so we cannot weaken the above hypothesis that  $X$  has less than  $\kappa$ -many factors. The equation  $Nt((\kappa+1)^\kappa) = \omega$  follows from a general theorem of Malykhin.

**Theorem 4.4.** [7] *Let  $X = \prod_{i \in I} X_i$  where each  $X_i$  has a minimal open cover of size two (e.g.,  $X_i$  is  $T_1$  and  $|X_i| \geq 2$ ). If  $\sup_{i \in I} w(X_i) \leq |I|$ , then  $Nt(X) = \omega$ .*

*Proof.* For each  $i \in I$ , let  $\{U_{i,0}, U_{i,1}\}$  be a minimal open cover of  $X_i$ . Since  $w(X) = \sup_{i \in I} w(X_i)$ , we may choose  $\mathcal{A}$  to be a base of  $X$  of size at most  $|I|$  and choose an injection  $f: \mathcal{A} \rightarrow I$ . Let  $\mathcal{B}$  denote the set of all nonempty sets of the form  $V \cap \pi_{f(V)}^{-1} [U_{f(V),j}]$  where  $V \in \mathcal{A}$  and  $j < 2$ . Since  $f$  is injective, the intersection of every infinite subset of  $\mathcal{B}$  has empty interior. Hence,  $\mathcal{B}$  is an  $\omega^{\text{op}}$ -like base of  $X$ .  $\square$

**Theorem 4.5.** *Let  $X$  be a product of countably many linearly ordered compacta. The following are equivalent.*

- (1)  $X$  is metrizable.
- (2)  $X$  has an  $\omega^{\text{op}}$ -like base.
- (3)  $X$  has an  $\omega_1^{\text{op}}$ -like base.
- (4)  $X$  is separable and has an  $\omega_1^{\text{op}}$ -like base.

*Proof.* By Theorem 3.1, (1) implies (2), which trivially implies (3). By Corollary 4.3, (3) implies (4). Finally, (4) implies (1) because if  $X$  is separable, then  $\pi(X) = \omega$ , so  $w(X) = \omega$  by Proposition 2.2.  $\square$

## 5. THE NOETHERIAN SPECTRUM OF THE COMPACT ORDERS

Theorem 4.5 implies that no linearly ordered compactum has Noetherian type  $\omega_1$ . What is the class of Noetherian types of linearly ordered compacta? We shall prove that an infinite cardinal  $\kappa$  is the Noetherian type of a linearly ordered compactum if and only if  $\kappa \neq \omega_1$  and  $\kappa$  is not weakly inaccessible.

**Theorem 5.1.** *Let  $\kappa$  be an uncountable cardinal and give  $\kappa+1$  the order topology. If  $\kappa$  is regular, then  $Nt(\kappa+1) = \kappa^+$ ; otherwise,  $Nt(\kappa+1) = \kappa$ .*

*Proof.* Using Corollary 4.3, the lower bounds on  $Nt(\kappa+1)$  are easy. We have  $d(\kappa+1) \geq \lambda$  for all regular  $\lambda \leq \kappa$ , so  $Nt(\kappa+1) > \lambda$  for all regular  $\lambda \leq \kappa$ . It follows that  $Nt(\kappa+1) \geq \kappa$  and  $Nt(\kappa+1) > \text{cf } \kappa$ . We can also prove these lower bounds directly using the Pressing Down Lemma. Let  $\mathcal{A}$  be a base of  $\kappa+1$  and let  $\lambda$  be a regular cardinal  $\leq \kappa$ . Let us show that  $\mathcal{A}$  is not  $\lambda^{\text{op}}$ -like. For every limit ordinal  $\alpha < \lambda$ , choose  $U_\alpha \in \mathcal{A}$  such that  $\alpha = \max U_\alpha$ ; choose  $\eta(\alpha) < \alpha$  such that  $[\eta(\alpha), \alpha] \subseteq U_\alpha$ . By the Pressing Down Lemma,  $\eta$  is constant on a stationary subset  $S$  of  $\lambda$ . Hence,  $\mathcal{A} \ni \{\eta(\min S) + 1\} \subseteq U_\alpha$  for all  $\alpha \in S$ ; hence,  $\mathcal{A}$  is not  $\lambda^{\text{op}}$ -like. Once again, it follows that  $Nt(\kappa+1) \geq \kappa$  and  $Nt(\kappa+1) > \text{cf } \kappa$ .

Trivially,  $Nt(\kappa+1) \leq w(\kappa+1)^+ = \kappa^+$ . Hence, it suffices to show that  $\kappa+1$  has a  $\kappa^{\text{op}}$ -like base if  $\kappa$  is singular. Suppose  $E \in [\kappa]^{<\kappa}$  is unbounded in  $\kappa$ . Let  $F$  be the set of limit points of  $E$  in  $\kappa+1$ . Define  $\mathcal{B}$  by

$$\mathcal{B} = \{(\beta, \alpha] : E \ni \beta < \alpha \in F \text{ or } \sup(E \cap \alpha) \leq \beta < \alpha \in \kappa \setminus F\}.$$

The set  $\mathcal{B}$  is a  $\kappa^{\text{op}}$ -like base of  $\kappa+1$ .  $\square$

**Definition 5.2.** Given a poset  $P$  with ordering  $\leq$ , let  $P^{\text{op}}$  denote the set  $P$  with ordering  $\geq$ .

**Theorem 5.3.** *Suppose  $\kappa$  is a singular cardinal. There then is a linearly ordered compactum with Noetherian type  $\kappa^+$ .*

*Proof.* Set  $\lambda = \text{cf } \kappa$  and  $X = \lambda^+ + 1$ . Partition the set of limit ordinals in  $\lambda^+$  into  $\lambda$ -many stationary sets  $\langle S_\alpha \rangle_{\alpha < \lambda}$ . Let  $\langle \kappa_\alpha \rangle_{\alpha < \lambda}$  be an increasing sequence of regular cardinals with supremum  $\kappa$ . For each  $\alpha < \lambda$  and  $\beta \in S_\alpha$ , set  $Y_\beta = (\kappa_\alpha + 1)^{\text{op}}$ . For each  $\alpha \in X \setminus \bigcup_{\beta < \lambda} S_\beta$ , set  $Y_\alpha = 1$ . Set  $Y = \bigcup_{\alpha \in X} \{\alpha\} \times Y_\alpha$  ordered lexicographically. We then have  $Nt(Y) \leq w(Y)^+ \leq |Y|^+ = \kappa^+$ . Hence, it suffices to show that  $Y$  has no  $\kappa^{\text{op}}$ -like base.

Seeking a contradiction, suppose  $\mathcal{A}$  is a  $\kappa^{\text{op}}$ -like base of  $Y$ . For each  $\alpha < \lambda$ , let  $\mathcal{U}_\alpha$  be the set of all  $U \in \mathcal{A}$  that have at least  $\kappa_\alpha$ -many supersets in  $\mathcal{A}$ . For all isolated points  $p$  of  $Y$ , there exists  $\alpha < \lambda$  such that  $\{p\} \notin \mathcal{U}_\alpha$ ; whence,  $p \notin \bigcup \mathcal{U}_\alpha$ . Since  $\langle \alpha + 1, 0 \rangle$  is isolated for all  $\alpha < \lambda^+$ , there exist  $\beta < \lambda$  and a set  $E$  of successor ordinals in  $\lambda^+$  such that  $|E| = \lambda^+$  and  $(E \times 1) \cap \bigcup \mathcal{U}_\beta = \emptyset$ . Let  $C$  be the closure of  $E$  in  $\lambda^+$ . The set  $C$  is closed unbounded; hence, there exists  $\gamma \in C \cap S_{\beta+1}$ . Set  $q = \langle \gamma, \kappa_{\beta+1} \rangle$ . We then have  $q \in \overline{E \times 1}$ ; hence,  $q \notin \bigcup \mathcal{U}_\beta$ . Since  $q$  has coinitality  $\kappa_{\beta+1}$ , any local base  $\mathcal{B}$  at  $q$  will contain an element  $U$  such that  $U$  has  $\kappa_\beta$ -many supersets in  $\mathcal{B}$ . Hence, there exists  $U \in \mathcal{U}_\beta$  such that  $q \in U$ ; hence,  $q \in \bigcup \mathcal{U}_\beta$ , which yields our desired contradiction.  $\square$

**Theorem 5.4.** *No linearly ordered compactum has weakly inaccessible Noetherian type. More generally, for every weakly inaccessible  $\kappa$ , products of fewer than  $\kappa$ -many linearly ordered compacta do not have Noetherian type  $\kappa$ .*

*Proof.* Suppose  $\kappa$  is weakly inaccessible,  $X$  is a product of fewer than  $\kappa$ -many linearly ordered compacta, and  $Nt(X) \leq \kappa$ . It suffices to prove  $Nt(X) < \kappa$ . By Corollary 4.3, we have  $d(X) < \kappa$ ; hence, each factor of  $X$  has  $\pi$ -weight less than  $\kappa$ ; hence,  $\pi(X) < \kappa$ . If  $w(X) \geq \kappa$ , then  $Nt(X) > \kappa$  by Proposition 2.2, in contradiction with our assumptions about  $X$ . Hence,  $w(X) < \kappa$ ; hence,  $Nt(X) \leq w(X)^+ < \kappa$ .  $\square$

## 6. THE LINDELÖF SPECTRUM

The spectrum of Noetherian types of Lindelöf linearly ordered topological spaces trivially includes the spectrum of Noetherian types of compact linearly ordered topological spaces. More interestingly, the inclusion is strict, as the next example shows.

**Example 6.1.** [12] Theorem 4.5 fails for Lindelöf linearly ordered topological spaces. Let  $X$  be  $(\omega_1 \times \mathbb{Z}) \cup (\{\omega_1\} \times \{0\})$  ordered lexicographically. The space  $X$  is Lindelöf and nonseparable and  $\{\langle \alpha, n \rangle : \alpha < \omega_1 \text{ and } n \in \mathbb{Z}\} \cup \{X \setminus (\alpha \times \mathbb{Z}) : \alpha < \omega_1\}$  is an  $\omega_1^{\text{op}}$ -like base of  $X$ . Moreover,  $X$  has no  $\omega^{\text{op}}$ -like base because every local base at  $\langle \omega_1, 0 \rangle$  includes a descending  $\omega_1$ -chain of neighborhoods. Thus,  $Nt(X) = \omega_1$ .

Easily generalizing this example, if  $\kappa$  is a regular cardinal and  $X$  is  $(\kappa \times \mathbb{Z}) \cup (\{\kappa\} \times \{0\})$  ordered lexicographically, then  $X$  is  $\kappa$ -compact and  $Nt(X) = \kappa$ .

A consequence of Lemma 4.1 is that Lindelöf linearly ordered topological spaces cannot have strongly inaccessible Noetherian type, just as in the compact case.

More generally, we have the following theorem, which is proved just as Theorem 5.4 was proved.

**Theorem 6.2.** *Suppose  $\kappa$  is a weakly inaccessible cardinal,  $|\lambda^{<\mu}| < \kappa$  for all  $\lambda < \kappa$ , and  $X$  is a product of fewer than  $\kappa$ -many  $\mu$ -compact GO-spaces. We then have  $Nt(X) \neq \kappa$ .*

*Proof.* Suppose that  $Nt(X) \leq \kappa$ . Let us show that  $Nt(X) < \kappa$ . By Lemma 4.1, we have  $d(X) < \kappa$ ; hence, each factor of  $X$  has  $\pi$ -weight less than  $\kappa$ ; hence,  $\pi(X) < \kappa$ . If  $w(X) \geq \kappa$ , then  $Nt(X) > \kappa$  by Proposition 2.2, in contradiction with our assumptions about  $X$ . Hence,  $w(X) < \kappa$ ; hence,  $Nt(X) \leq w(X)^+ < \kappa$ .  $\square$

**Corollary 6.3.** *If  $\kappa$  is strongly inaccessible, then the class of Noetherian types of  $\mu$ -compact GO-spaces excludes  $\kappa$  if and only if  $\mu < \kappa$ .*

*Proof.* “If”: Theorem 6.2. “Only if”: The above generalization of Example 6.1.  $\square$

On the other hand, it is consistent (relative to the consistency of an inaccessible), that some Lindelöf linearly ordered topological space has weakly inaccessible Noetherian type. To show this, we first force  $2^{\aleph_0} \geq \kappa$  where  $\kappa$  is weakly inaccessible (say, by adding  $\kappa$ -many Cohen reals). Next, we construct the desired linear order in this forcing extension using the following theorem.

**Theorem 6.4.** *If  $\kappa$  is a weakly inaccessible and  $2^{\aleph_0} \geq \kappa$ , then there is a Lindelöf linear order  $Z$  such that  $Nt(Z) = \kappa$ .*

*Proof.* Let  $B$  be a Bernstein subset of  $X = [0, 1]$ , i.e.,  $B$  includes some point in  $P$  and misses some point in  $P$ , for all perfect  $P \subseteq X$ . Let  $f: B \rightarrow \kappa$  be surjective. For each  $x \in B$ , set  $Y_x = \omega^{\text{op}} + \omega_{f(x)} + \omega$ , which is Lindelöf. For each  $x \in X \setminus B$ , set  $Y_x = \{0\}$ . Set  $Z = \bigcup_{x \in X} (\{x\} \times Y_x)$  ordered lexicographically. First, let us show that  $Z$  is Lindelöf. Let  $\mathcal{U}$  be an open cover of  $Z$ . For every  $x \in X \setminus B$ ,  $\langle x, 0 \rangle$  has neighborhoods  $O_x$  and  $U_x$  such that  $\mathcal{U} \ni U_x \supseteq O_x = \bigcup_{a < b < c} (\{b\} \times Y_b)$  where  $a, c \in (X \cap \mathbb{Q}) \cup \{\pm\infty\}$ . Therefore, there is a countable  $D \subseteq X$  such that  $\{O_x : x \in D\}$  covers  $(X \setminus B) \times \{0\}$ . Set  $\mathcal{V} = \{U_x : x \in D\}$  and  $C = \{x \in X : Y_x \not\subseteq \bigcup_{x \in D} O_x\}$ . The set  $C$  is closed in  $X$  and a subset of the Bernstein set  $B$ , so  $C$  is countable. Therefore,  $\bigcup_{x \in C} (\{x\} \times Y_x)$  is Lindelöf; hence, it is covered by a countable  $\mathcal{W} \subseteq \mathcal{U}$ , making  $\mathcal{V} \cup \mathcal{W}$  a countable subcover of  $\mathcal{U}$ .

Finally, let us show that  $Nt(Z) = \kappa$ . For every  $\alpha < \kappa$  and  $x \in f^{-1}[\{\alpha + 1\}]$ ,  $Y_x$  has a point with cofinality  $\omega_{\alpha+1}$ , so  $Nt(Z) \geq \omega_{\alpha+1}$ . Therefore, it suffices to construct a  $\kappa^{\text{op}}$ -like base of  $Z$ . Let  $\mathcal{A}$  denote the countable set of all sets of the form  $\bigcup_{a < b < c} Y_b$  where  $a, c \in (X \cap \mathbb{Q}) \cup \{\pm\infty\}$ , which includes a local base at  $\langle x, 0 \rangle$  for every  $x \in X \setminus B$ . Since each  $Y_x$  for  $x \in B$  has no maximum or minimum, we can combine  $\mathcal{A}$  with a copy of a base  $\mathcal{B}_x$  of  $Y_x$  for each  $x \in B$  in order to produce a base  $\mathcal{B}$  of  $Z$ . We may choose each  $\mathcal{B}_x$  to have size less than  $\kappa$ , so  $\mathcal{B}$  must be  $\kappa^{\text{op}}$ -like.  $\square$

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