

SPLITTING FAMILIES AND THE NOETHERIAN TYPE OF

$$\beta\omega \setminus \omega$$

DAVID MILOVICH

ABSTRACT. Extending some results of Malykhin, we prove several independence results about base properties of $\beta\omega \setminus \omega$ and its powers, especially the Noetherian type $Nt(\beta\omega \setminus \omega)$, the least κ for which $\beta\omega \setminus \omega$ has a base that is κ -like with respect to containment. For example, $Nt(\beta\omega \setminus \omega)$ is at least \mathfrak{s} , but can consistently be ω_1 , \mathfrak{c} , \mathfrak{c}^+ , or strictly between ω_1 and \mathfrak{c} . $Nt(\beta\omega \setminus \omega)$ is also consistently less than the additivity of the meager ideal. $Nt(\beta\omega \setminus \omega)$ is closely related to the existence of special kinds of splitting families.

1. INTRODUCTION

Definition 1.1. Given a cardinal κ , define a poset to be κ -like (κ^{op} -like) if no element is above (below) κ -many elements. Define a poset to be *almost* κ^{op} -like if it has a κ^{op} -like dense subset.

In the context of families of subsets of a topological space, we will always implicitly order by inclusion. We are particularly interested in κ^{op} -like bases, π -bases, local bases, and local π -bases of the space ω^* of nonprincipal ultrafilters on ω . Recall that a local base (local π -base) at a point in a space is a family of open neighborhoods of that point (family of nonempty open subsets) such that every neighborhood of the point contains an element of the family; a base (π -base) of a space is family of open sets that contains local bases (local π -bases) at every point. See Engelking [9] for the more background on bases and their cousins. Also recall the following basic cardinal functions. For more about these functions, see Juhász [12].

Definition 1.2. Given a space X , let the *weight* of X , or $w(X)$, be the least $\kappa \geq \omega$ such that X has a base of size at most κ . Given $p \in X$, let the *character* of p , or $\chi(p, X)$, be the least $\kappa \geq \omega$ such that there is a local base at p of size at most κ . Let the character of X , or $\chi(X)$, be the supremum of the characters of its points. Analogously define π -*weight* and π -*character*, respectively denoting them using π and $\pi\chi$.

Now consider the following order-theoretic parallels.

Definition 1.3. Given a space X , let the *Noetherian type* of X , or $Nt(X)$, be the least $\kappa \geq \omega$ such that X has a base that is κ^{op} -like. Given $p \in X$, let the *local Noetherian type* of p , or $\chi Nt(p, X)$, be the least $\kappa \geq \omega$ such that there is a κ^{op} -like local base at p . Let the local Noetherian type of X , or $\chi Nt(X)$, be the supremum of the local Noetherian types of its points. Analogously define *Noetherian π -type* and *local Noetherian π -type*, respectively denoting them using πNt and $\pi\chi Nt$.

Date: March 26, 2008.

Support provided by an NSF graduate fellowship.

Noetherian type and Noetherian π -type were introduced by Peregudov [16]. Let ω^* denote the space of nonprincipal ultrafilters on ω . Malykhin [15] proved that MA implies $\pi Nt(\omega^*) = \mathfrak{c}$ and CH implies $Nt(\omega^*) = \mathfrak{c}$. We extend these results by investigating $Nt(\omega^*)$, $\pi Nt(\omega^*)$, $\chi Nt(\omega^*)$, and $\pi\chi Nt(\omega^*)$ as cardinal characteristics of the continuum. For background on such cardinals, see Blass [7]. We also examine the sequence $\langle Nt((\omega^*)^{1+\alpha}) \rangle_{\alpha \in \mathcal{O}_n}$.

Definition 1.4. Let \mathfrak{b} denote the minimum of $|\mathcal{F}|$ where \mathcal{F} ranges over the subsets of ω^ω that have no upper bound in ω^ω with respect to eventual domination.

Definition 1.5. A *tree* π -base of a space X is a π -base that is a tree when ordered by containment. Let \mathfrak{h} be the minimum of the set of heights of tree π -bases of ω^* .

Balcar, Pelant, and Simon [1] proved that tree π -bases of ω^* exist, and that $\mathfrak{h} \leq \min\{\mathfrak{b}, \text{cf } \mathfrak{c}\}$. They also proved that the above definition of \mathfrak{h} is equivalent to the more common definition of \mathfrak{h} as the distributivity number of $[\omega]^\omega$ ordered by \subseteq^* .

Definition 1.6. Given $x, y \in [\omega]^\omega$, we say that x *splits* y if $|y \cap x| = |y \setminus x| = \omega$. Let \mathfrak{r} be the minimum value of $|A|$ where A ranges over the subsets of $[\omega]^\omega$ such that no $x \in [\omega]^\omega$ splits every $y \in A$. Let \mathfrak{s} be the minimum value of $|A|$ where A ranges over the subsets of $[\omega]^\omega$ such that every $x \in [\omega]^\omega$ is split by some $y \in A$.

It is known that $\mathfrak{b} \leq \mathfrak{r}$ and $\mathfrak{h} \leq \mathfrak{s}$. (See Theorems 3.8 and 6.9 of [7].)

Clearly, $Nt(\omega^*) \leq w(\omega^*)^+ = \mathfrak{c}^+$. We will show that also $\pi\chi Nt(\omega^*) = \omega$ and $\pi Nt(\omega^*) = \mathfrak{h}$ and $\mathfrak{s} \leq Nt(\omega^*)$. Furthermore, $Nt(\omega^*)$ can consistently be \mathfrak{c} , \mathfrak{c}^+ , or any regular κ satisfying $2^{<\kappa} = \mathfrak{c}$. Also, $Nt(\omega^*) = \omega_1$ is relatively consistent with any values of \mathfrak{b} and \mathfrak{c} . The relations $\omega_1 < \mathfrak{b} = \mathfrak{s} = Nt(\omega^*) < \mathfrak{c}$ and $\omega_1 = \mathfrak{b} = \mathfrak{s} < Nt(\omega^*) < \mathfrak{c}$ are also each consistent. We also prove some relations between \mathfrak{r} and $Nt(\omega^*)$, as well as some consistency results about the local Noetherian type of points in ω^* .

2. BASIC RESULTS

The following proposition is essentially due to Peregudov (see Lemma 1 of [16]).

Proposition 2.1. *Suppose a point p in a space X satisfies $\pi\chi(p, X) < \text{cf } \kappa \leq \kappa \leq \chi(p, X)$. Then $Nt(X) > \kappa$.*

Proof. Let \mathcal{A} be a base of X . Let \mathcal{U}_0 and \mathcal{V}_0 be, respectively, a local π -base at p of size at most $\pi\chi(p, X)$ and a local base at p of size $\chi(p, X)$. For each element of \mathcal{U}_0 , choose a subset in \mathcal{A} , thereby producing local π -base \mathcal{U} at p that is a subset of \mathcal{A} of size at most $\pi\chi(p, X)$. Similarly, for each element of \mathcal{V}_0 , choose a smaller neighborhood of p in \mathcal{A} , thereby producing a local base \mathcal{V} at p that is a subset of \mathcal{A} of size $\chi(p, X)$. Every element of \mathcal{V} contains an element of \mathcal{U} . Hence, some element of \mathcal{U} is contained in κ -many elements of \mathcal{V} ; hence, \mathcal{A} is not κ^{op} -like. \square

Definition 2.2. For all $x \in [\omega]^\omega$, set $x^* = \{p \in \omega^* : p \in x\}$.

Theorem 2.3. *It is relatively consistent with any value of \mathfrak{c} satisfying $\text{cf } \mathfrak{c} > \omega_1$ that $Nt(\omega^*) = \mathfrak{c}^+$.*

Proof. We may assume $\text{cf } \mathfrak{c} > \omega_1$. By Exercise A10 on p. 289 of Kunen [14], there is a ccc generic extension $V[G]$ such that $\check{\mathfrak{c}} = \mathfrak{c}^{V[G]}$ and, in $V[G]$, there exists $p \in \omega^*$

such that $\chi(p, \omega^*) = \omega_1$. Henceforth work in $V[G]$. Let φ be a bijection from ω^2 to ω . Define $\psi: \omega^* \rightarrow \omega^*$ by

$$x \mapsto \{E \subseteq \omega : \{m < \omega : \{n < \omega : \varphi(m, n) \in E\} \in p\} \in x\}.$$

Since $\pi\chi(p, \omega^*) \leq \chi(p, \omega^*) = \omega_1$, there exists $\langle E_\alpha \rangle_{\alpha < \omega_1} \in ([\omega]^\omega)^{\omega_1}$ such that every neighborhood of p contains E_α^* for some $\alpha < \omega_1$. Hence, for all $x \in \omega^*$, every neighborhood of $\psi(x)$ contains $(\varphi''(\{m\} \times E_\alpha))^*$ for some $m < \omega$ and $\alpha < \omega_1$; whence, $\pi\chi(\psi(x), \omega^*) = \omega_1$. Since ψ is easily verified to be a topological embedding, $\chi(x, \omega^*) \leq \chi(\psi(x), \omega^*)$ for all $x \in \omega^*$. By a result of Pospíšil [17], there exists $q \in \omega^*$ such that $\chi(q, \omega^*) = \mathfrak{c}$. Hence, $\pi\chi(\psi(q), \omega^*) = \omega_1$ and $\chi(\psi(q), \omega^*) = \mathfrak{c}$. By Proposition 2.1, $Nt(\omega^*) > \chi(\psi(q), \omega^*) = \mathfrak{c}$. \square

Definition 2.4. Given $n < \omega$, let \mathfrak{ss}_n (\mathfrak{ss}_ω) denote the least cardinal κ for which there exists a sequence $\langle f_\alpha \rangle_{\alpha < \kappa}$ of functions on ω each with range contained in n (each with finite range) such that for all $I \in [\kappa]^\kappa$ and $x \in [\omega]^\omega$ there exists $\alpha \in I$ such that f_α is not eventually constant on x . (The notation \mathfrak{ss} was chosen with the phrase ‘‘supersplitting number’’ in mind.) Note that if such an $\langle f_\alpha \rangle_{\alpha < \kappa}$ does not exist for any $\kappa \leq \mathfrak{c}$, then \mathfrak{ss}_n (\mathfrak{ss}_ω) is by definition equal to \mathfrak{c}^+ .

Clearly $\mathfrak{ss}_n \geq \mathfrak{ss}_{n+1} \geq \mathfrak{ss}_\omega$ for all $n < \omega$. Moreover, since $\text{cf } \mathfrak{c} > \omega$, we have $\mathfrak{ss}_\omega = \mathfrak{ss}_n$ for some $n < \omega$. However, for any particular $n \in \omega \setminus 2$, it is not clear whether ZFC proves $\mathfrak{ss}_\omega = \mathfrak{ss}_n$.

Definition 2.5. Given $\lambda \geq \kappa \geq \omega$ and a space X , a $\langle \lambda, \kappa \rangle$ -splitter of X is a sequence $\langle \mathcal{F}_\alpha \rangle_{\alpha < \lambda}$ of finite open covers of X such that, for all $I \in [\lambda]^\kappa$ and $\langle U_\alpha \rangle_{\alpha \in I} \in \prod_{\alpha \in I} \mathcal{F}_\alpha$, the interior of $\bigcap_{\alpha \in I} U_\alpha$ is empty.

Lemma 2.6. *Suppose X is a compact space with a base \mathcal{A} of size at most $w(X)$ such that $U \cap V \in \mathcal{A} \cup \{\emptyset\}$ for all $U, V \in \mathcal{A}$. If $\kappa \leq w(X)$ and X has a $\langle w(X), \kappa \rangle$ -splitter, then \mathcal{A} contains a κ^{op} -like base of X . Hence, $Nt(\omega^*) \leq \mathfrak{ss}_\omega$.*

Proof. Set $\lambda = w(X)$ and let $\langle \mathcal{F}_\alpha \rangle_{\alpha < \lambda}$ be a $\langle \lambda, \kappa \rangle$ -splitter of X . For each $\alpha < \lambda$, the cover \mathcal{F}_α is refined by a finite subcover of \mathcal{A} ; hence, we may assume $\mathcal{F}_\alpha \subseteq \mathcal{A}$. Let $\mathcal{A} = \{U_\alpha : \alpha < \lambda\}$. For each $\alpha < \lambda$, set $\mathcal{B}_\alpha = \{U_\alpha \cap V : V \in \mathcal{F}_\alpha\}$. Set $\mathcal{B} = \bigcup_{\alpha < \lambda} \mathcal{B}_\alpha \setminus \{\emptyset\}$. Then \mathcal{B} is easily seen to be a base of X and a κ^{op} -like subset of \mathcal{A} . \square

Lemma 2.7. *Let X be a compact space without isolated points and let $\omega \leq \kappa \leq \lambda \leq \min_{p \in X} \chi(p, X)$. If X has no $\langle \lambda, \kappa \rangle$ -splitter, then $Nt(X) > \kappa$.*

Proof. Let \mathcal{A} be a base of X . Construct a sequence $\langle \mathcal{F}_\alpha \rangle_{\alpha < \lambda}$ of finite subcovers of \mathcal{A} as follows. Suppose we have $\alpha < \lambda$ and $\langle \mathcal{F}_\beta \rangle_{\beta < \alpha}$. For each $p \in X$, choose $V_p \in \mathcal{A}$ such that $p \in V_p \notin \bigcup_{\beta < \alpha} \mathcal{F}_\beta$. Let \mathcal{F}_α be a finite subcover of $\{V_p : p \in X\}$. Then $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for all $\alpha < \beta < \lambda$. Suppose X has no $\langle \lambda, \kappa \rangle$ -splitter. Then choose $I \in [\lambda]^\kappa$ and $\langle U_\alpha \rangle_{\alpha \in I} \in \prod_{\alpha \in I} \mathcal{F}_\alpha$ such that $\bigcap_{\alpha \in I} U_\alpha$ has nonempty interior. Then there exists $W \in \mathcal{A}$ such that $W \subseteq \bigcap_{\alpha \in I} U_\alpha$. Thus, \mathcal{A} is not κ^{op} -like. \square

Definition 2.8. Let \mathfrak{u} denote the minimum of the set of characters of points in ω^* . Let $\pi\mathfrak{u}$ denote the minimum of the set of π -characters of points in ω^* .

By a theorem of Balcar and Simon [2], $\pi\mathfrak{u} = \mathfrak{r}$.

Theorem 2.9. *Suppose $\mathfrak{u} = \mathfrak{c}$. Then $Nt(\omega^*) = \mathfrak{ss}_\omega$.*

Proof. By Lemma 2.6, $Nt(\omega^*) \leq \mathfrak{ss}_\omega$. Suppose $\kappa \leq \mathfrak{c}$. Since every finite open cover of ω^* is refined by a finite, pairwise disjoint, clopen cover, ω^* has a $\langle \mathfrak{c}, \kappa \rangle$ -splitter if and only if $\mathfrak{ss}_\omega \leq \kappa$. Hence, $Nt(\omega^*) \geq \mathfrak{ss}_\omega$ by Lemma 2.7. \square

Lemma 2.10. *Suppose $\mathfrak{r} = \mathfrak{c}$. Then $\mathfrak{ss}_2 \leq \mathfrak{c}$.*

Proof. Let $\langle x_\alpha \rangle_{\alpha < \mathfrak{c}}$ enumerate $[\omega]^\omega$. Construct $\langle y_\alpha \rangle_{\alpha < \mathfrak{c}} \in ([\omega]^\omega)^\mathfrak{c}$ as follows. Given $\alpha < \mathfrak{c}$ and $\langle y_\beta \rangle_{\beta < \alpha}$, choose y_α such that y_α splits every element of $\{x_\alpha\} \cup \{y_\beta : \beta < \alpha\}$. Suppose $I \in [\mathfrak{c}]^\mathfrak{c}$ and $\alpha < \mathfrak{c}$. Then x_α is split by y_β for all $\beta \in I \setminus \alpha$. Thus, $\langle \{y_\alpha, \omega \setminus y_\alpha\} \rangle_{\alpha < \mathfrak{c}}$ witnesses $\mathfrak{ss}_2 \leq \mathfrak{c}$. \square

Theorem 2.11. *The cardinals \mathfrak{r} and $Nt(\omega^*)$ are related as follows.*

- (1) *If $\mathfrak{r} = \mathfrak{c}$, then $Nt(\omega^*) = \mathfrak{ss}_\omega \leq \mathfrak{c}$.*
- (2) *If $\mathfrak{r} < \mathfrak{c}$, then $Nt(\omega^*) \geq \mathfrak{c}$.*
- (3) *If $\mathfrak{r} < \text{cf } \mathfrak{c}$, then $Nt(\omega^*) = \mathfrak{c}^+$.*

Proof. Statement (1) follows from Lemma 2.10, Theorem 2.9, and $\pi\mathfrak{u} = \mathfrak{r}$. The proof of Theorem 2.3 shows how to construct $p \in \omega^*$ such that $\pi\chi(p, \omega^*) = \pi\mathfrak{u} = \mathfrak{r}$ and $\chi(p, \omega^*) = \mathfrak{c}$. Hence, (2) and (3) follow from Proposition 2.1. \square

Definition 2.12. A subset A of $[\omega]^\omega$ has the *strong finite intersection property* (SFIP) if the intersection of every finite subset of A is infinite. Given $A \subseteq [\omega]^\omega$ with the SFIP, define the *Booth forcing for A* to be $[\omega]^{<\omega} \times [A]^{<\omega}$ ordered by $\langle \sigma_0, F_0 \rangle \leq \langle \sigma_1, F_1 \rangle$ if and only if $F_0 \supseteq F_1$ and $\sigma_1 \subseteq \sigma_0 \subseteq \sigma_1 \cup \bigcap F_1$. Define a *generic pseudointersection* of A to be $\bigcup_{\langle \sigma, F \rangle \in G} \sigma$ where G is a generic filter of $[\omega]^{<\omega} \times [A]^{<\omega}$.

Theorem 2.13. *For all cardinals κ satisfying $\kappa > \text{cf } \kappa > \omega$, it is consistent that $\mathfrak{r} = \mathfrak{u} = \text{cf } \kappa$ and $Nt(\omega^*) = \mathfrak{ss}_2 = \mathfrak{c} = \kappa$.*

Proof. Assuming GCH in the ground model, construct a finite support iteration $\langle \mathbb{P}_\alpha \rangle_{\alpha < \kappa}$ as follows. First choose some $U_0 \in \omega^*$. Then suppose we have $\alpha < \kappa$ and \mathbb{P}_α and $\Vdash_\alpha U_\alpha \in \omega^*$. Let $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ where \mathbb{Q}_α is a \mathbb{P}_α -name for the Booth forcing for U_α . Let x_α be a $\mathbb{P}_{\alpha+1}$ -name for a generic pseudointersection of U_α added by \mathbb{Q}_α ; let $U_{\alpha+1}$ be a $\mathbb{P}_{\alpha+1}$ -name for an element of ω^* containing $U_\alpha \cup \{x_\alpha\}$. For limit $\alpha < \kappa$, let $U_\alpha = \bigcup_{\beta < \alpha} U_\beta$.

Let $\langle \eta_\alpha \rangle_{\alpha < \text{cf } \kappa}$ be an increasing sequence of ordinals with supremum κ . Then $\{x_{\eta_\alpha} : \alpha < \text{cf } \kappa\}$ is forced to generate an ultrafilter in $V^{\mathbb{P}^\kappa}$. Hence, $\Vdash_\kappa \mathfrak{r} \leq \mathfrak{u} \leq \text{cf } \kappa < \kappa = \mathfrak{c}$. Therefore, by Lemma 2.6 and Theorem 2.11, it suffices to show that $\Vdash_\kappa \mathfrak{ss}_2 \leq \kappa$. Every nontrivial finite support iteration of infinite length adds a Cohen real. Hence, we may choose for each $\alpha < \kappa$ a $\mathbb{P}_{\omega(\alpha+1)}$ -name y_α for an element of $[\omega]^\omega$ that is Cohen over $V^{\mathbb{P}^{\omega\alpha}}$. Then every name S for the range of a cofinal subsequence of $\langle y_\alpha \rangle_{\alpha < \kappa}$ is such that

$$\Vdash_\kappa \forall z \in [\omega]^\omega \exists w \in S \text{ } w \text{ splits } z.$$

Hence, $\langle y_\alpha \rangle_{\alpha < \kappa}$ witnesses that $\Vdash_\kappa \mathfrak{ss}_2 \leq \kappa$. \square

Theorem 2.14. *$Nt(\omega^*) \geq \mathfrak{s}$.*

Proof. Suppose $Nt(\omega^*) = \kappa < \mathfrak{s}$. Since $Nt(\omega^*) < \mathfrak{c}$, we have $\mathfrak{r} = \mathfrak{c}$ by Theorem 2.11. Hence, $\mathfrak{u} = \mathfrak{c}$. By Theorem 2.9, it suffices to show that $\mathfrak{ss}_\omega > \kappa$. Suppose $\langle f_\alpha \rangle_{\alpha < \mathfrak{c}}$ is a sequence of functions on ω with finite range and $I \in [\mathfrak{c}]^\kappa$. Since $\kappa < \mathfrak{s}$, there exists $x \in [\omega]^\omega$ such that f_α is eventually constant on x for all $\alpha \in I$. Thus, $\mathfrak{ss}_\omega > \kappa$. \square

Lemma 2.15. *Let κ be a cardinal and let P and Q be mutually dense subsets of a common poset. Then P is almost κ^{op} -like if and only if Q is.*

Proof. Suppose D is a κ^{op} -like dense subset of P . Then it suffices to construct a κ^{op} -like dense subset of Q . Define a partial map f from $|D|^+$ to Q as follows. Set $f_0 = \emptyset$. Suppose $\alpha < |D|^+$ and we have constructed a partial map f_α from α to Q . Set $E = \{d \in D : d \not\leq q \text{ for all } q \in \text{ran } f_\alpha\}$. If $E = \emptyset$, then set $f_{\alpha+1} = f_\alpha$. Otherwise, choose $q \in Q$ such that $q \leq e$ for some $e \in E$ and let $f_{\alpha+1}$ be the smallest function extending f_α such that $f_{\alpha+1}(\alpha) = q$. For limit ordinals $\gamma \leq |D|^+$, set $f_\gamma = \bigcup_{\alpha < \gamma} f_\alpha$. Set $f = f_{|D|^+}$.

Let us show that $\text{ran } f$ is a κ^{op} -like. Suppose otherwise. Then there exists $q \in \text{ran } f$ and an increasing sequence $\langle \xi_\alpha \rangle_{\alpha < \kappa}$ in $\text{dom } f$ such that $q \leq f(\xi_\alpha)$ for all $\alpha < \kappa$. By the way we constructed f , there exists $\langle d_\alpha \rangle_{\alpha < \kappa} \in D^\kappa$ such that $f(\xi_\beta) \leq d_\beta \neq d_\alpha$ for all $\alpha < \beta < \kappa$. Choose $p \in P$ such that $p \leq q$. Then choose $d \in D$ such that $d \leq p$. Then $d \leq d_\beta \neq d_\alpha$ for all $\alpha < \beta < \kappa$, which contradicts that D is κ^{op} -like. Therefore, $\text{ran } f$ is κ^{op} -like.

Finally, let us show that $\text{ran } f$ is a dense subset of Q . Suppose $q \in Q$. Choose $p \in P$ such that $p \leq q$. Then choose $d \in D$ such that $d \leq p$. By the way we constructed f , there exists $r \in \text{ran } f$ such that $r \leq d$; hence, $r \leq q$. \square

Theorem 2.16. $\pi Nt(\omega^*) = \mathfrak{h}$.

Proof. First, we show that $\pi Nt(\omega^*) \leq \mathfrak{h}$. Let \mathcal{A} be a tree π -base of ω^* such that \mathcal{A} has height \mathfrak{h} with respect to containment. Then \mathcal{A} is clearly \mathfrak{h}^{op} -like. To show that $\mathfrak{h} \leq \pi Nt(\omega^*)$, let \mathcal{A} be as above and let \mathcal{B} be a $\pi Nt(\omega^*)^{\text{op}}$ -like π -base of ω^* . Then \mathcal{A} and \mathcal{B} are mutually dense; hence, by Lemma 2.15, \mathcal{A} contains a $\pi Nt(\omega^*)^{\text{op}}$ -like π -base \mathcal{C} of ω^* . Since \mathcal{C} is also a tree π -base, it has height at most $\pi Nt(\omega^*)$. Hence, $\mathfrak{h} \leq \pi Nt(\omega^*)$. \square

Corollary 2.17. *If $\mathfrak{h} = \mathfrak{c}$, then $\pi Nt(\omega^*) = Nt(\omega^*) = \mathfrak{ss}_2 = \mathfrak{c}$.*

Proof. Suppose $\mathfrak{h} = \mathfrak{c}$. Then $\mathfrak{r} = \mathfrak{c}$ because $\mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{c}$. Hence, by Theorem 2.16, Theorem 2.11, and Lemma 2.10, $\mathfrak{c} \leq \pi Nt(\omega^*) \leq Nt(\omega^*) = \mathfrak{ss}_\omega \leq \mathfrak{ss}_2 \leq \mathfrak{c}$. \square

3. MODELS OF $Nt(\omega^*) = \omega_1$

Adding \mathfrak{c} -many Cohen reals collapses \mathfrak{ss}_2 to ω_1 . By Lemma 2.6, it therefore also collapses $Nt(\omega^*)$ to ω_1 . The same result holds for random reals and Hechler reals.

Theorem 3.1. *Suppose $\kappa^\omega = \kappa$ and $\mathbb{P} = \mathcal{B}(2^\kappa)/\mathcal{I}$ where $\mathcal{B}(2^\kappa)$ is the Borel algebra of the product space 2^κ and \mathcal{I} is either the meager ideal or the null ideal (with respect to the product measure). (In other words, \mathbb{P} adds κ -many Cohen reals or κ -many random reals in the usual way.) Then $\mathbb{1}_{\mathbb{P}} \Vdash \omega_1 = \mathfrak{ss}_2$.*

Proof. Working in the generic extension $V[G]$, we have $\kappa = \mathfrak{c}$ and a sequence $\langle x_\alpha \rangle_{\alpha < \kappa}$ in $[\omega]^\omega$ such that $V[G] = V[\langle x_\alpha \rangle_{\alpha < \kappa}]$ and, if $E \in \mathcal{P}(\kappa) \cap V$ and $\alpha \in \kappa \setminus E$, then x_α is Cohen or random over $V[\langle x_\beta \rangle_{\beta \in E}]$. (See [13] for a proof.) Suppose $I \in [\kappa]^{\omega_1}$ and $y \in [\omega]^\omega$. Then $y \in V[\langle x_\alpha \rangle_{\alpha \in J}]$ for some $J \in [\kappa]^\omega \cap V$; hence, x_α splits y for all $\alpha \in I \setminus J$. Thus, $\langle \{x_\alpha, \omega \setminus x_\alpha\} \rangle_{\alpha < \kappa}$ witnesses $\mathfrak{ss}_2 = \omega_1$. \square

Definition 3.2. Let \mathfrak{d} denote the minimum of the cardinalities of subsets of ω^ω that are cofinal with respect to eventual domination.

Corollary 3.3. *Every transitive model of ZFC has a ccc forcing extension that preserves \mathfrak{b} , \mathfrak{d} , and \mathfrak{c} , and collapses \mathfrak{ss}_2 to ω_1 .*

Proof. Add \mathfrak{c} -many random reals to the ground model. Then every element of ω^ω in the extension is eventually dominated by an element of ω^ω in the ground model; hence, \mathfrak{b} , \mathfrak{d} , and \mathfrak{c} are preserved by this forcing, while \mathfrak{ss}_2 becomes ω_1 . \square

Definition 3.4. We say that a transfinite sequence $\langle x_\alpha \rangle_{\alpha < \eta}$ of subsets of ω is *eventually splitting* if for all $y \in [\omega]^\omega$ there exists $\alpha < \eta$ such that for all $\beta \in \eta \setminus \alpha$ the set x_β splits y .

Theorem 3.5. *Let $\kappa = \kappa^\omega$. Then $\mathfrak{ss}_2 = \omega_1$ is forced by the κ -long finite support iteration of Hechler forcing.*

Proof. Let \mathbb{P} be the κ -long finite support iteration of Hechler forcing. Let G be a generic filter of \mathbb{P} . For each $\alpha < \kappa$, let g_α be the generic dominating function added at stage α ; set $x_\alpha = \{n < \omega : g_\alpha(n) \text{ is even}\}$. Suppose $p \in G$ and I and y are names such that p forces $I \in [\kappa]^\omega$ and $y \in [\omega]^\omega$. Choose $q \in G$ and a name h such that $q \leq p$ and q forces h to be an increasing map from ω_1 to I . For each $\alpha < \omega_1$, set $E_\alpha = \{\beta < \kappa : q \nVdash h(\alpha) \neq \check{\beta}\}$; let k_α be a surjection from ω to E_α . Let $q \geq r \in G$ and $n < \omega$ and $\gamma \leq \kappa$ and J be a name such that r forces $J \in [\omega_1]^\omega$ and $\text{sup ran } h = \check{\gamma}$ and $h(\alpha) = k_\alpha(n)$ for all $\alpha \in J$. Set $F = \{k_\alpha(n) : \alpha < \omega_1\} \cap \gamma$; let j be the order isomorphism from some ordinal η to F . Then $\text{cf } \eta = \text{cf } \gamma = \omega_1$. For all $\alpha < \kappa$, the set x_α is Cohen over $V[\langle g_\beta \rangle_{\beta < \alpha}]$; hence, $\langle x_{j(\alpha)} \rangle_{\alpha < \eta}$ is eventually splitting in $V[\langle g_\alpha \rangle_{\alpha < \gamma}]$. By a result of Baumgartner and Dordal [5], $\langle x_{j(\alpha)} \rangle_{\alpha < \eta}$ is also eventually splitting in $V[G]$. Choose $\beta < \eta$ such that $x_{j(\alpha)}$ splits y_G for all $\alpha \in \eta \setminus \beta$. Then there exist $s \in G$ and $\alpha \in \gamma \setminus j(\beta)$ such that $r \geq s \Vdash \check{\alpha} \in h \text{``} J$. Hence, $\alpha \in I_G$ and x_α splits y_G . Thus, $\langle \{x_\alpha, \omega \setminus x_\alpha\} \rangle_{\alpha < \kappa}$ witnesses $\mathfrak{ss}_2 = \omega_1$ in $V[G]$. \square

Definition 3.6. Let $\text{add}(\mathcal{B})$ denote the additivity of the ideal of meager sets of reals.

It is known that $\text{add}(\mathcal{B}) \leq \mathfrak{b}$ and that it is consistent that $\text{add}(\mathcal{B}) < \mathfrak{b}$. (See 5.4 and 11.7 of [7] and 7.3.D of [4]).

Corollary 3.7. *If $\kappa = \text{cf } \kappa > \omega$, then it is consistent that $\mathfrak{ss}_2 = \omega_1$ and $\text{add}(\mathcal{B}) = \mathfrak{c} = \kappa$.*

Proof. Starting with GCH in the ground model, perform a κ -long finite support iteration of Hechler forcing. This forces $\text{add}(\mathcal{B}) = \mathfrak{c} = \kappa$ (see 11.6 of [7]). By Theorem 3.5, this also forces $\mathfrak{ss}_2 = \omega_1$. \square

4. MODELS OF $\omega_1 < \text{Nt}(\omega^*) < \mathfrak{c}$

To prove the consistency of $\omega_1 < \text{Nt}(\omega^*) < \mathfrak{c}$, we employ generalized iteration of forcing along posets as defined by Groszek and Jech [10]. We will only use finite support iterations along well-founded posets. For simplicity, we limit our definition of generalized iterations to this special case.

Definition 4.1. Suppose X is a well-founded poset and \mathbb{P} a forcing order consisting of functions on X . Given any $x \in X$, partial map f on X , and down-set Y of X , set $\mathbb{P} \upharpoonright Y = \{p \upharpoonright Y : p \in \mathbb{P}\}$, $X \upharpoonright x = \{y \in X : y < x\}$, $X \upharpoonright_{\leq} x = \{y \in X : y \leq x\}$, $\mathbb{P} \upharpoonright x = \mathbb{P} \upharpoonright (X \upharpoonright x)$, $\mathbb{P} \upharpoonright_{\leq} x = \mathbb{P} \upharpoonright (X \upharpoonright_{\leq} x)$, $f \upharpoonright x = f \upharpoonright (X \upharpoonright x)$, and $f \upharpoonright_{\leq} x = f \upharpoonright$

($X \upharpoonright_{\leq} x$). Then \mathbb{P} is a *finite support iteration along X* if there exists a sequence $\langle \mathbb{Q}_x \rangle_{x \in X}$ satisfying the following conditions for all $x \in X$ and all $p, q \in \mathbb{P}$.

- (1) $\mathbb{P} \upharpoonright x$ is a finite support iteration along $X \upharpoonright x$.
- (2) \mathbb{Q}_x is a $(\mathbb{P} \upharpoonright x)$ -name for a forcing order.
- (3) $\mathbb{P} \upharpoonright_{\leq} x = \{p \cup \{\langle x, q \rangle\} : \langle p, q \rangle \in (\mathbb{P} \upharpoonright x) * \mathbb{Q}_x\}$.
- (4) $\mathbb{1}_{\mathbb{P}} \upharpoonright x \Vdash \mathbb{1}_{\mathbb{P}}(x) = \mathbb{1}_{\mathbb{Q}_x}$.
- (5) \mathbb{P} is the set of functions r on X for which $r \upharpoonright_{\leq} y \in \mathbb{P} \upharpoonright_{\leq} y$ for all $y \in X$ and $\mathbb{1}_{\mathbb{P} \upharpoonright z} \Vdash r(z) = \mathbb{1}_{\mathbb{Q}_z}$ for all but finitely many $z \in X$.
- (6) $p \leq q$ if and only if $p \upharpoonright y \leq q \upharpoonright y$ and $p \upharpoonright y \Vdash p(y) \leq q(y)$ for all $y \in X$.

Given a finite support iteration \mathbb{P} along X and $x \in X$ and a filter G of \mathbb{P} , set $G_x = \{p(x) : p \in G\}$, $G \upharpoonright x = \{p \upharpoonright x : p \in G\}$, and $G \upharpoonright_{\leq} x = \{p \upharpoonright_{\leq} x : p \in G\}$. Given any down-set Y of X , set $G \upharpoonright Y = \{p \upharpoonright Y : p \in G\}$.

Remark. If \mathbb{P} is a finite support iteration along a well-founded poset X with down-set Y , then $\mathbb{P} \upharpoonright Y$ is an iteration along Y , and $\mathbb{1}_{\mathbb{P} \upharpoonright Y} = \mathbb{1}_{\mathbb{P}} \upharpoonright Y$.

Definition 4.2. Suppose \mathbb{P} is a finite support iteration along a well-founded poset X with down-sets Y and Z such that $Y \subseteq Z$. Then there is a complete embedding $j_Y^Z: \mathbb{P} \upharpoonright Y \rightarrow \mathbb{P} \upharpoonright Z$ given by $j_Y^Z(p) = p \cup (\mathbb{1}_{\mathbb{P}} \upharpoonright Z \setminus Y)$ for all $p \in \mathbb{P} \upharpoonright Y$. This embedding naturally induces an embedding of the class of $(\mathbb{P} \upharpoonright Y)$ -names, which in turn naturally induces an embedding of the class of atomic formulae in the $(\mathbb{P} \upharpoonright Y)$ -forcing language. Let j_Y^Z also denote these embeddings.

Proposition 4.3. *Suppose \mathbb{P} , Y , and Z are as in the above definition, and φ is an atomic formula in the $(\mathbb{P} \upharpoonright Y)$ -forcing language. Then, for all $p \in \mathbb{P} \upharpoonright Z$, we have $p \Vdash j_Y^Z(\varphi)$ if and only if $p \upharpoonright Y \Vdash \varphi$.*

Proof. If $p \upharpoonright Y \Vdash \varphi$, then $p \leq j_Y^Z(p \upharpoonright Y) \Vdash j_Y^Z(\varphi)$. Conversely, suppose $p \upharpoonright Y \not\Vdash \varphi$. Then we may choose $q \leq p \upharpoonright Y$ such that $q \Vdash \neg\varphi$. Hence, $j_Y^Z(q) \Vdash \neg j_Y^Z(\varphi)$. Set $r = q \cup (p \upharpoonright Z \setminus Y)$. Then $j_Y^Z(q) \geq r \leq p$; hence, $p \not\Vdash j_Y^Z(\varphi)$. \square

Lemma 4.4. *Suppose \mathbb{P} is a finite support iteration along a well-founded poset X and x is a maximal element of X . Set $Y = X \setminus \{x\}$. Then there is a dense embedding $\phi: \mathbb{P} \rightarrow (\mathbb{P} \upharpoonright Y) * j_{X \upharpoonright x}^Y(\mathbb{Q}_x)$ given by $\phi(p) = \langle p \upharpoonright Y, j_{X \upharpoonright x}^Y(p(x)) \rangle$. Hence, if G is a \mathbb{P} -generic filter, then G_x is $(\mathbb{Q}_x)_{G \upharpoonright x}$ -generic over $V[G \upharpoonright Y]$.*

Proof. First, let us show that ϕ is an order embedding. Suppose $r, s \in \mathbb{P}$. Then $r \leq s$ if and only if $r \upharpoonright Y \leq s \upharpoonright Y$ and $r \upharpoonright x \Vdash r(x) \leq s(x)$. Also, $\phi(r) \leq \phi(s)$ if and only if $r \upharpoonright Y \leq s \upharpoonright Y$ and $r \upharpoonright Y \Vdash j_{X \upharpoonright x}^Y(r(x)) \leq s(x)$. By Proposition 4.3, $r \upharpoonright Y \Vdash j_{X \upharpoonright x}^Y(r(x)) \leq s(x)$ if and only if $r \upharpoonright x \Vdash r(x) \leq s(x)$; hence, $r \leq s$ if and only if $\phi(r) \leq \phi(s)$.

Finally, let us show that $\text{ran } \phi$ is dense. Suppose $\langle p, q \rangle \in (\mathbb{P} \upharpoonright Y) * j_{X \upharpoonright x}^Y(\mathbb{Q}_x)$. Then there exist $r \leq p$ and $s \in \text{dom}(j_{X \upharpoonright x}^Y(\mathbb{Q}_x))$ such that $r \Vdash s = q \in j_{X \upharpoonright x}^Y(\mathbb{Q}_x)$. Hence, $\langle r, s \rangle \leq \langle p, q \rangle$. Also, s is a $(j_{X \upharpoonright x}^Y(\mathbb{P} \upharpoonright x))$ -name; hence, there exists a $(\mathbb{P} \upharpoonright x)$ -name t such that $j_{X \upharpoonright x}^Y(t) = s$. Hence, $r \Vdash j_{X \upharpoonright x}^Y(t \in \mathbb{Q}_x)$; hence, $r \upharpoonright x \Vdash t \in \mathbb{Q}_x$. Hence, $r \cup \{\langle x, t \rangle\} \in \mathbb{P}$ and $\phi(r \cup \{\langle x, t \rangle\}) = \langle r, s \rangle$. Thus, $\text{ran } \phi$ is dense. \square

Remark. Proposition 4.3 and Lemma 4.4 and their proofs remain valid for arbitrary iterations along posets as defined in [10].

Lemma 4.5. *Let \mathbb{P} be a forcing order, A a subset of $[\omega]^\omega$ with the SFIP, \mathbb{Q} the Booth forcing for A , x a \mathbb{Q} -name for a generic pseudointersection of A , and B a*

\mathbb{P} -name such that $\mathbb{1}_{\mathbb{P}}$ forces $\check{A} \subseteq B \subseteq [\omega]^\omega$ and forces B to have the SFIP. Let i and j be the canonical embeddings, respectively, of \mathbb{P} -names and \mathbb{Q} -names into $(\mathbb{P} * \check{\mathbb{Q}})$ -names. Then $\mathbb{1}_{\mathbb{P} * \check{\mathbb{Q}}}$ forces $i(B) \cup \{j(x)\}$ to have the SFIP.

Proof. Seeking a contradiction, suppose $r_0 = \langle p_0, \langle \sigma, F \rangle \rangle \in \mathbb{P} * \check{\mathbb{Q}}$ and $n < \omega$ and $p_0 \Vdash H \in [B]^{<\omega}$ and $r_0 \Vdash j(x) \cap \bigcap i(H) \subseteq \check{n}$. Then p_0 forces $\check{F} \cup H \subseteq B$, which is forced to have the SFIP; hence, there exist $p_1 \leq p_0$ and $m \in \omega \setminus n$ such that $p_1 \Vdash \check{m} \in \bigcap (\check{F} \cup H)$. Set $r_1 = \langle p_1, \langle \sigma \cup \{m\}, F \rangle \rangle$. Then $r_0 \geq r_1 \Vdash \check{m} \in j(x) \cap \bigcap i(H)$, contradicting how we chose r_0 . \square

Lemma 4.6. *Suppose \mathbb{P} and \mathbb{Q} are forcing orders such that \mathbb{P} is ccc and \mathbb{Q} has property (K). Then $\mathbb{1}_{\mathbb{P}}$ forces $\check{\mathbb{Q}}$ to have property (K).*

Proof. Suppose the lemma fails. Then there exist $p \in \mathbb{P}$ and f such that $p \Vdash f \in \check{\mathbb{Q}}^{\omega_1}$ and $p \Vdash \forall J \in [\omega_1]^{\omega_1} \exists \alpha, \beta \in J f(\alpha) \perp f(\beta)$. For each $\alpha < \omega_1$, choose $p_\alpha \leq p$ and $q_\alpha \in \mathbb{Q}$ such that $p_\alpha \Vdash f(\alpha) = \check{q}_\alpha$. Then there exists $I \in [\omega_1]^{\omega_1}$ such that $q_\alpha \not\leq q_\beta$ for all $\alpha, \beta \in I$. Let J be the \mathbb{P} -name $\{\langle \check{\alpha}, p_\alpha \rangle : \alpha \in I\}$. Then $p \Vdash \forall \alpha, \beta \in J f(\alpha) = \check{q}_\alpha \not\leq \check{q}_\beta = f(\beta)$. Hence, $p \Vdash |J| \leq \omega$. Since \mathbb{P} is ccc, there exists $\alpha \in I$ such that $p \Vdash J \subseteq \check{\alpha}$. But this contradicts $p \geq p_\alpha \Vdash \check{\alpha} \in J$. \square

Lemma 4.7. *Suppose \mathbb{P} is a finite support iteration along a well-founded poset X and $\mathbb{1}_{\mathbb{P}} \upharpoonright x$ forces \mathbb{Q}_x to have property (K) for all $x \in X$. Then \mathbb{P} has property (K).*

Proof. We may assume the lemma holds whenever X is replaced by a poset of lesser height. Let $I \in [\mathbb{P}]^{\omega_1}$. We may assume $\{\text{supp}(p) : p \in I\}$ is a Δ -system; let σ be its root. Set $Y_0 = \bigcup_{x \in \sigma} X \upharpoonright x$. Then $\mathbb{P} \upharpoonright Y_0$ has property (K). Let $n = |\sigma \setminus Y_0|$ and $\langle x_i \rangle_{i < n}$ biject from n to $\sigma \setminus Y_0$. Set $Y_{i+1} = Y_i \cup \{x_i\}$ for all $i < n$. Suppose $i < n$ and $\mathbb{P} \upharpoonright Y_i$ has property (K). By Lemma 4.6, $\mathbb{1}_{\mathbb{P} \upharpoonright Y_i}$ forces $j_{X \upharpoonright x_i}^{Y_i}(\mathbb{Q}_{x_i})$ to have property (K). Hence, $\mathbb{P} \upharpoonright Y_{i+1}$ has property (K), for it densely embeds into $\mathbb{P} \upharpoonright Y_i * j_{X \upharpoonright x_i}^{Y_i}(\mathbb{Q}_{x_i})$ by Lemma 4.4. By induction, $\mathbb{P} \upharpoonright Y_n$ has property (K); hence, there exists $J \in [I]^{\omega_1}$ such that $p \upharpoonright Y_n \not\leq q \upharpoonright Y_n$ for all $p, q \in J$. Fix $p, q \in J$ and choose r such that $r \leq p \upharpoonright Y_n$ and $r \leq q \upharpoonright Y_n$. Set $s = r \cup (p \upharpoonright \text{supp}(p) \setminus Y_n) \cup (q \upharpoonright \text{supp}(q) \setminus Y_n)$ and $t = s \cup (\mathbb{1}_{\mathbb{P}} \upharpoonright X \setminus \text{dom } s)$. Then $t \leq p, q$. \square

Lemma 4.8. *Suppose $\text{cf } \kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. Then there exists a κ -like, κ -directed, well-founded poset Ξ with cofinality and cardinality λ .*

Proof. Let $\{x_\alpha : \alpha < \lambda\}$ biject from λ to $[\lambda]^{<\kappa}$. Construct $\langle y_\alpha \rangle_{\alpha < \lambda} \in ([\lambda]^{<\kappa})^\lambda$ as follows. Given $\alpha < \lambda$ and $\langle y_\beta \rangle_{\beta < \alpha}$, choose $\xi_\alpha \in \lambda \setminus \bigcup_{\beta < \alpha} y_\beta$ and set $y_\alpha = x_\alpha \cup \{\xi_\alpha\}$. Let Ξ be $\{y_\alpha : \alpha < \lambda\}$ ordered by inclusion. Then Ξ is cofinal with $[\lambda]^{<\kappa}$; hence, Ξ is κ -directed and has cofinality λ . Also, Ξ is well-founded because $\langle y_\alpha \rangle_{\alpha < \lambda}$ is nondecreasing. Finally, Ξ is κ -like because for all $I \in [\lambda]^\kappa$ we have $|\bigcup_{\alpha \in I} y_\alpha| \geq |\{\xi_\alpha : \alpha \in I\}| = \kappa$; whence, $\{y_\alpha : \alpha \in I\}$ has no upper bound in $[\lambda]^{<\kappa}$. \square

Definition 4.9. A point q in a space X is a P_κ -point if every intersection of fewer than κ -many neighborhoods of q contains a neighborhood of q .

Definition 4.10. For all $x, y \subseteq \omega$, define $x \subseteq^* y$ as $|x \setminus y| < \omega$. Let \mathfrak{p} denote the minimum value of $|A|$ where A ranges over the subsets of $[\omega]^\omega$ that have SFIP yet have no pseudointersection.

Remark. It easily seen that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{h}$.

Theorem 4.11. *Suppose $\omega_1 \leq \text{cf } \kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. Then there is a property (K) forcing extension in which*

$$\mathfrak{p} = \pi \text{Nt}(\omega^*) = \text{Nt}(\omega^*) = \mathfrak{ss}_2 = \mathfrak{b} = \kappa \leq \lambda = \mathfrak{c}.$$

Moreover, in this extension ω^* has P_κ -points; whence, $\max_{q \in \omega^*} \chi \text{Nt}(q, \omega^*) = \kappa$.

Proof. Let Ξ be as in Lemma 4.8. Let $\langle \sigma_\alpha \rangle_{\alpha < \lambda}$ biject from λ to Ξ . Let $\langle \langle \zeta_\alpha, \eta_\alpha \rangle \rangle_{\alpha < \lambda}$ biject from λ to λ^2 . Given $\alpha < \lambda$ and $\langle \tau_{\zeta_\beta, \eta_\beta} \rangle_{\beta < \alpha} \in \Xi^\alpha$, choose $\tau_{\zeta_\alpha, \eta_\alpha} \in \Xi$ such that $\sigma_{\zeta_\alpha} < \tau_{\zeta_\alpha, \eta_\alpha} \not\leq \tau_{\zeta_\beta, \eta_\beta}$ for all $\beta < \alpha$. We may so choose $\tau_{\zeta_\alpha, \eta_\alpha}$ because Ξ is directed and has cofinality λ .

Let us construct a finite support iteration \mathbb{P} along Ξ . Since Ξ is well-founded, we may define \mathbb{Q}_σ in terms of $\mathbb{P} \upharpoonright \sigma$ for each $\sigma \in \Xi$. Suppose $\sigma \in \Xi$ and, for all $\tau < \sigma$, we have $|\mathbb{P} \upharpoonright \tau| < \kappa$ and $\mathbb{1}_{\mathbb{P} \upharpoonright \tau}$ forces \mathbb{Q}_τ to have property (K). Then $\mathbb{P} \upharpoonright \sigma$ has property (K) by Lemma 4.7, and hence is ccc. Moreover, $|\mathbb{P} \upharpoonright \sigma| < \kappa$ because $\mathbb{P} \upharpoonright \sigma$ is a finite support iteration along $\Xi \upharpoonright \sigma$ and $|\Xi \upharpoonright \sigma| < \kappa$. Hence, $\mathbb{1}_{\mathbb{P} \upharpoonright \sigma} \Vdash |\mathfrak{c}^{<\kappa}| \leq ((\kappa^\omega)^{<\kappa}) \leq \lambda$. Let \mathcal{E}_σ be a $(\mathbb{P} \upharpoonright \sigma)$ -name for the set of all E in the $(\mathbb{P} \upharpoonright \sigma)$ -generic extension for which $E \in [[\omega]^\omega]^{<\kappa}$ and E has the SFIP. Then we may choose a $(\mathbb{P} \upharpoonright \sigma)$ -name f_σ such that $\mathbb{1}_{\mathbb{P} \upharpoonright \sigma}$ forces f_σ to be a surjection from λ to \mathcal{E}_σ . We may assume we are given corresponding f_τ for all $\tau < \sigma$. If there exist $\alpha, \beta < \lambda$ such that $\sigma = \tau_{\alpha, \beta}$, then let \mathbb{Q}_σ be a $(\mathbb{P} \upharpoonright \sigma)$ -name for $\mathbb{Q}'_\sigma \times \text{Fn}(\omega, 2)$ where \mathbb{Q}'_σ is a $(\mathbb{P} \upharpoonright \sigma)$ -name for the Booth forcing for $f_{\sigma_\alpha}(\beta)$. If there are no such α and β , then let \mathbb{Q}_σ be a $(\mathbb{P} \upharpoonright \sigma)$ -name for a singleton poset. Then $\mathbb{1}_{\mathbb{P} \upharpoonright \sigma}$ forces \mathbb{Q}_σ to have property (K). Also, we may assume $|\mathbb{Q}_\sigma| < \kappa$. Hence, $|\mathbb{P} \upharpoonright \sigma| < \kappa$.

By induction, $|\mathbb{P} \upharpoonright \sigma| < \kappa$ and $\mathbb{1}_{\mathbb{P} \upharpoonright \sigma}$ forces \mathbb{Q}_σ to have property (K) for all $\sigma \in \Xi$. Hence, \mathbb{P} has property (K) by Lemma 4.7, and hence is ccc. Also, since $|\Xi| \leq \lambda$ and \mathbb{P} is a finite support iteration, $|\mathbb{P}| \leq \lambda$. Let G be a \mathbb{P} -generic filter. Then $\mathfrak{c}^{V[G]} \leq \lambda^\omega = \lambda$. Moreover, $\mathfrak{c}^{V[G]} \geq \lambda$ because \mathbb{P} adds λ -many Cohen reals.

By Theorem 2.16 and Lemma 2.6, it suffices to show that $\mathfrak{b}^{V[G]} \leq \kappa \leq \mathfrak{p}^{V[G]}$, that $\mathfrak{ss}_2^{V[G]} \leq \kappa$, and that some $q \in (\omega^*)^{V[G]}$ is a P_κ -point. First, we prove $\kappa \leq \mathfrak{p}^{V[G]}$. Suppose $E \in ([[\omega]^\omega]^{<\kappa})^{V[G]}$ and E has the SFIP. Then there exists $\alpha < \lambda$ such that $E \in V[G \upharpoonright \sigma_\alpha]$ because Ξ is κ -directed. Hence, there exists $\beta < \lambda$ such that $(f_{\sigma_\alpha})_{G \upharpoonright \sigma_\alpha}(\beta) = E$. Hence, E has a pseudointersection in $V[G \upharpoonright \tau_{\alpha, \beta}]$. Thus, $\kappa \leq \mathfrak{p}^{V[G]}$.

Second, let us show that $\mathfrak{b}^{V[G]} \leq \kappa$. For each $\alpha < \kappa$, let u_α be the increasing enumeration of the Cohen real added by the $\text{Fn}(\omega, 2)$ factor of $\mathbb{Q}_{\tau_{0, \alpha}}$. Then it suffices to show that $\{u_\alpha : \alpha < \kappa\}$ is unbounded in $(\omega^\omega)^{V[G]}$. Suppose $v \in (\omega^\omega)^{V[G]}$. Then there exists $\sigma \in \Xi$ such that $v \in V[G \upharpoonright \sigma]$. Since Ξ is κ -like, there exists $\alpha < \kappa$ such that $\tau_{0, \alpha} \not\leq \sigma$. By Lemma 4.4, u_α enumerates a real Cohen generic over $V[G \upharpoonright \sigma]$; hence, u_α is not eventually dominated by v .

Third, let us prove $\mathfrak{ss}_2^{V[G]} \leq \kappa$. For each $\alpha < \lambda$, let x_α be the Cohen real added by the $\text{Fn}(\omega, 2)$ factor of $\mathbb{Q}_{\tau_{0, \alpha}}$. Suppose $I \in ([\lambda]^\kappa)^{V[G]}$ and $y \in ([\omega]^\omega)^{V[G]}$. Then there exists $\sigma \in \Xi$ such that $y \in V[G \upharpoonright \sigma]$. Since Ξ is κ -like, there exists $\alpha \in I$ such that $\tau_{0, \alpha} \not\leq \sigma$. By Lemma 4.4, x_α is Cohen generic over $V[G \upharpoonright \sigma]$, and therefore splits y . Thus, $\langle \{x_\alpha, \omega \setminus x_\alpha\} \rangle_{\alpha < \lambda}$ witnesses $\mathfrak{ss}_2^{V[G]} \leq \kappa$.

Finally, let us construct a P_κ -point $q \in (\omega^*)^{V[G]}$. Let \sqsubseteq be an extension of the ordering of Ξ to a well-ordering of Ξ . For each $\sigma \in \Xi$, set $Y_\sigma = \{\tau \in \Xi : \tau \sqsubset \sigma\}$. Set $\rho = \min_{\sqsubseteq} \Xi$ and choose $U_\rho \in (\omega^*)^V$. Suppose $\tau \in \Xi$ and σ is a final predecessor of τ with respect to \sqsubseteq and $U_\sigma \in (\omega^*)^{V[G \upharpoonright Y_\sigma]}$. If there are no $\alpha, \beta < \lambda$ such that

$\sigma = \tau_{\alpha, \beta}$ and $(f_{\sigma_\alpha})_{G \upharpoonright \sigma_\alpha}(\beta) \subseteq U_\sigma$, then choose $U_\tau \in (\omega^*)^{V[G \upharpoonright Y_\tau]}$ such that $U_\tau \supseteq U_\sigma$. Now suppose such α and β exist. Let v_σ be the pseudointersection of $(f_{\sigma_\alpha})_{G \upharpoonright \sigma_\alpha}(\beta)$ added by \mathbb{Q}'_σ .

By Lemmas 4.4 and 4.5, $U_\sigma \cup \{v_\sigma\}$ has the SFIP; hence, we may choose $U_\tau \in (\omega^*)^{V[G \upharpoonright Y_\tau]}$ such that $U_\tau \supseteq U_\sigma \cup \{v_\sigma\}$. For $\tau \in \Xi$ that are limit points with respect to \sqsubseteq , choose $U_\tau \in (\omega^*)^{V[G \upharpoonright Y_\tau]}$ such that $U_\tau \supseteq \bigcup_{\sigma \sqsubset \tau} U_\sigma$; set $q = \bigcup_{\tau \in \Xi} U_\tau$. Then, arguing as in the proof of $\kappa \leq \mathfrak{p}^{V[G]}$, we have that q is a P_κ -point in $(\omega^*)^{V[G]}$. \square

The forcing extension of Theorem 4.11 can be modified to satisfy $\mathfrak{b} = \mathfrak{s} < Nt(\omega^*) < \mathfrak{c}$.

Definition 4.12. Given a class \mathcal{J} of posets and a cardinal κ , let $MA(\kappa; \mathcal{J})$ denote the statement that, given any $\mathbb{P} \in \mathcal{J}$ and fewer than κ -many dense subsets of \mathbb{P} , there is a filter of \mathbb{P} intersecting each of these dense sets. We may replace \mathcal{J} with a descriptive term for \mathcal{J} when there is no ambiguity. For example, $MA(\mathfrak{c}; \text{ccc})$ is Martin's axiom.

Theorem 4.13. *Suppose $\omega_1 < \text{cf } \kappa = \kappa \leq \lambda = \lambda^{<\kappa}$. Then there is a property (K) forcing extension in which*

$$\omega_1 = \pi Nt(\omega^*) = \mathfrak{b} = \mathfrak{s} < Nt(\omega^*) = \mathfrak{s}\mathfrak{s}_2 = \kappa \leq \lambda = \mathfrak{c}.$$

Proof. Let \mathbb{P} be as in the proof of Theorem 4.11. Set $\mathbb{R} = \mathbb{P} \times \text{Fn}(\omega_1, 2)$, which has property (K) because \mathbb{P} does. Let K be a generic filter of \mathbb{R} . Let π_0 and π_1 be the natural coordinate projections on \mathbb{R} ; let π_0 and π_1 also denote their respective natural extensions to the class of \mathbb{R} -names. Set $G = \pi_0 \text{``} K$ and $H = \pi_1 \text{``} K$. Then $\mathfrak{c}^{V[K]} = \lambda$ clearly holds. Adding ω_1 -many Cohen reals to any model of ZFC forces $\mathfrak{b} = \mathfrak{s} = \omega_1$, and $\pi Nt(\omega^*) = \mathfrak{h} \leq \mathfrak{b}$, so $\pi Nt(\omega^*)^{V[K]} = \mathfrak{b}^{V[K]} = \mathfrak{s}^{V[K]} = \omega_1$.

For each $\alpha < \lambda$, let x_α be the Cohen real added by the $\text{Fn}(\omega, 2)$ factor of $\mathbb{Q}_{\tau_{0, \alpha}}$. Suppose $I \in ([\lambda]^\kappa)^{V[K]}$ and $y \in ([\omega]^\omega)^{V[K]}$. Then there exists $\sigma \in \Xi$ such that $y \in V[(G \upharpoonright \sigma) \times H]$. Since Ξ is κ -like, there exists $\alpha \in I$ such that $\tau_{0, \alpha} \not\leq \sigma$. By Lemma 4.4, x_α is Cohen generic over $V[G \upharpoonright \sigma]$; hence, x_α is Cohen generic over $V[(G \upharpoonright \sigma) \times H]$ and therefore splits y . Thus, $\langle \{x_\alpha, \omega \setminus x_\alpha\} \rangle_{\alpha < \lambda}$ witnesses $\mathfrak{s}\mathfrak{s}_2^{V[K]} \leq \kappa$.

Therefore, it suffices to show that $Nt(\omega^*)^{V[K]} \geq \kappa$. Suppose $\mu < \kappa$ and \mathcal{A} is an \mathbb{R} -name for a base of ω^* . Choose an \mathbb{R} -name q for an element of ω^* with character λ . Let f be a name for an injection from λ into \mathcal{A} such that $q \in \bigcap \text{ran } f$. Let g be a name for an element of $([\omega]^\omega)^\lambda$ such that $q \in g(\alpha)^* \subseteq f(\alpha)$ for all $\alpha < \lambda$. For each $\alpha < \lambda$, let u_α be a name for $g(\alpha)$ such that $u_\alpha = \{\{\check{n}\} \times A_{\alpha, n} : n < \omega\}$ where each $A_{\alpha, n}$ is a countable antichain of \mathbb{R} . Since $\max\{\omega_1, \mu\} < \lambda$, there exist $\xi < \omega_1$ and $J \in [\lambda]^\mu$ such that $\text{ran } \pi_1(u_\alpha) \subseteq \text{Fn}(\xi, 2)$ for all $\alpha \in J$. It suffices to show that $\{(u_\alpha)_K : \alpha \in J\}$ has a pseudointersection in $V[K]$.

For each $\alpha \in J$, set $v_\alpha = \{\langle \check{n}, r \rangle : \langle \check{n}, \langle p, r \rangle \rangle \in u_\alpha \text{ and } p \in G\}$. Set $H_0 = H \cap \text{Fn}(\xi, 2)$. By Bell's Theorem [6], $MA(\mathfrak{p}; \sigma\text{-centered})$ is a theorem of ZFC. Hence, $V[G]$ satisfies $MA(\kappa; \sigma\text{-centered})$. By an argument of Baumgartner and Tall communicated by Roitman [18], adding a single Cohen real preserves $MA(\kappa; \sigma\text{-centered})$. Since Booth forcing for $\{(v_\alpha)_{H_0} : \alpha \in J\}$ is σ -centered, $\{(v_\alpha)_{H_0} : \alpha \in J\}$, which is equal to $\{(u_\alpha)_K : \alpha \in J\}$, has a pseudointersection in $V[G \times H_0]$. \square

5. LOCAL NOETHERIAN TYPE AND π -TYPE

Definition 5.1. For every infinite cardinal κ , let $u(\kappa)$ denote the space of uniform ultrafilters on κ .

Dow and Zhou [8] proved that there is a point in ω^* that (along with satisfying some additional properties) has an ω^{op} -like local base. We present a simpler construction of an ω^{op} -like local base which also naturally generalizes to every $u(\kappa)$. This construction is essentially due to Isbell [11], who was interested in actual intersections as opposed to pseudointersections.

Definition 5.2. Given cardinals $\lambda \geq \kappa \geq \omega$ and a point p in a space X , a *local $\langle \lambda, \kappa \rangle$ -splitter* is a set \mathcal{U} of λ -many open neighborhoods of p such that p is not in the interior of $\bigcap \mathcal{V}$ for any $\mathcal{V} \in [\mathcal{U}]^\kappa$.

Lemma 5.3. *Every poset P is almost $|P|^{\text{op}}$ -like.*

Proof. Let $\kappa = |P|$ and let $\langle p_\alpha \rangle_{\alpha < \kappa}$ biject from κ to P . Define a partial map $f: \kappa \rightarrow P$ as follows. Suppose $\alpha < \kappa$ and we have a partial map $f_\alpha: \alpha \rightarrow P$. If $\text{ran } f_\alpha$ is dense in P , then set $f_{\alpha+1} = f_\alpha$. Otherwise, set $\beta = \min\{\delta < \kappa : p_\delta \not\geq q \text{ for all } q \in \text{ran } f_\alpha\}$ and set $f_{\alpha+1} = f_\alpha \cup \{\langle \alpha, p_\beta \rangle\}$. For limit ordinals $\gamma \leq \kappa$, set $f_\gamma = \bigcup_{\alpha < \gamma} f_\alpha$. Set $f = f_\kappa$. Then f is nonincreasing; hence, $\text{ran } f$ is κ^{op} -like. Moreover, $\text{ran } f$ is dense in P . \square

Lemma 5.4. *Suppose X is a space with a point p at which there is no finite local base. Then $\chi \text{Nt}(p, X)$ is the least $\kappa \geq \omega$ for which there is a local $\langle \chi(p, X), \kappa \rangle$ -splitter at p . Moreover, if $\lambda > \chi(p, X)$, then p does not have a local $\langle \lambda, \kappa \rangle$ -splitter at p for any $\kappa < \lambda$ or $\kappa \leq \text{cf } \lambda$.*

Proof. By Lemma 5.3, $\chi(p, X) \geq \chi \text{Nt}(p, X)$; hence, a $\chi \text{Nt}(p, X)^{\text{op}}$ -like local base at p (which necessarily has size $\chi(p, X)$) is a local $\langle \chi(p, X), \chi \text{Nt}(p, X) \rangle$ -splitter at p . To show the converse, let $\lambda = \chi(p, X)$ and let $\langle U_\alpha \rangle_{\alpha < \lambda}$ be a sequence of open neighborhoods of p . Let $\{V_\alpha : \alpha < \lambda\}$ be a local base at p . For each $\alpha < \lambda$, choose $W_\alpha \in \{V_\beta : \beta < \lambda\}$ such that $W_\alpha \subseteq U_\alpha \cap V_\alpha$. Then $\{W_\alpha : \alpha < \lambda\}$ is a local base at p . Let $\kappa < \chi \text{Nt}(p, X)$. Then there exist $\alpha < \lambda$ and $I \in [\lambda]^\kappa$ such that $W_\alpha \subseteq \bigcap_{\beta \in I} W_\beta$. Hence, p is in the interior of $\bigcap_{\beta \in I} U_\beta$. Hence, $\{U_\alpha : \alpha < \lambda\}$ is not a local $\langle \lambda, \kappa \rangle$ -splitter at p .

To prove the second half of the lemma, suppose $\lambda > \chi(p, X)$ and \mathcal{A} is a set of λ -many open neighborhoods of p . Let \mathcal{B} be a local base at p of size $\chi(p, X)$. Then, for all $\kappa < \lambda$ and $\kappa \leq \text{cf } \lambda$, there exist $U \in \mathcal{B}$ and $\mathcal{C} \in [\mathcal{A}]^\kappa$ such that $U \subseteq \bigcap \mathcal{C}$. Hence, \mathcal{A} is not a local $\langle \lambda, \kappa \rangle$ -splitter at p . \square

Theorem 5.5. *For each $\kappa \geq \omega$, there exists $p \in u(\kappa)$ such that $\chi \text{Nt}(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$.*

Proof. Let A be an independent family of subsets of κ of size 2^κ . Set $B = \bigcup_{F \in [A]^\omega} \{x \subseteq \kappa : \forall y \in F \ |x \setminus y| < \kappa\}$. Since A is independent, we may extend A to an ultrafilter p on κ such that $p \cap B = \emptyset$. For each $x \subseteq \kappa$, set $x^* = \{q \in u(\kappa) : x \in q\}$. Then $\{x^* : x \in A\}$ is a local $\langle 2^\kappa, \omega \rangle$ -splitter at p . Since $\chi(p, u(\kappa)) \leq 2^\kappa$, it follows from Lemma 5.4 that $\chi \text{Nt}(p, u(\kappa)) = \omega$ and $\chi(p, u(\kappa)) = 2^\kappa$. \square

Definition 5.6. Let \mathfrak{a} denote the minimum of the cardinalities of infinite, maximal almost disjoint subfamilies of $[\omega]^\omega$. Let \mathfrak{i} denote the minimum of the cardinalities of infinite, maximal independent subfamilies of $[\omega]^\omega$.

It is known that $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{r} \leq \mathfrak{i} \leq \mathfrak{d} \leq \mathfrak{s}$. (See 8.4, 8.12, 8.13 and 3.3 of [7].) Because of Kunen's result that $\mathfrak{a} = \aleph_1$ in the Cohen model (see VIII.2.3 of [14]), it is consistent that $\mathfrak{a} < \mathfrak{r}$. Also, Shelah [20] has constructed a model of $\mathfrak{r} \leq \mathfrak{u} < \mathfrak{a}$.

In ZFC, the best upper bound of $\chi Nt(\omega^*)$ of which we know is \mathfrak{c} by Lemma 5.3. We will next prove Theorem 5.10, which implies that, except for \mathfrak{c} and possibly of \mathfrak{c} , all of the cardinal characteristics of the continuum with definitions included in Blass [7] can consistently be simultaneously strictly less than $\chi Nt(\omega^*)$.

Lemma 5.7. *Suppose κ , λ , and μ are regular cardinals and $\kappa \leq \lambda > \mu$. Then $(\kappa \times \lambda)^{\text{op}}$ is not almost μ^{op} -like.*

Proof. Let I be a cofinal subset of $\kappa \times \lambda$. Then it suffices to show that I is not μ -like. If $\kappa = \lambda$, then I is not μ -like because it is λ -directed. Suppose $\kappa < \lambda$. Then there exists $\alpha < \kappa$ such that $|I \cap (\{\alpha\} \times \lambda)| = \lambda$; hence, I has an increasing λ -sequence; hence, I is not μ -like. \square

Lemma 5.8. *Given any infinite independent subfamily I of $[\omega]^\omega$, there exists $J \subseteq [\omega]^\omega$ such that if x is a generic pseudointersection of J then $I \cup \{x\}$ is independent, but $I \cup \{x, y\}$ is not independent for any $y \in [\omega]^\omega \cap V \setminus I$.*

Proof. See Exercise A12 on page 289 of Kunen [14]. \square

Definition 5.9. We say a P_κ -point in a space is *simple* if it has a local base of order type κ^{op} .

Theorem 5.10. *Suppose $\omega_1 \leq \text{cf } \kappa = \kappa \leq \text{cf } \lambda = \lambda = \lambda^{<\kappa}$. Then there is a property (K) forcing extension satisfying $\mathfrak{p} = \mathfrak{a} = \mathfrak{i} = \mathfrak{u} = \kappa \leq \lambda = \chi Nt(\omega^*) = \mathfrak{c}$.*

Proof. We will construct a finite support iteration $\langle \mathbb{P}_\alpha \rangle_{\alpha \leq \lambda\kappa}$ where $\lambda\kappa$ denotes the ordinal product of λ and κ . It suffices to ensure that the iteration is at every stage property (K) and of size at most λ , and that $V^{\mathbb{P}^{\lambda\kappa}}$ satisfies $\max\{\mathfrak{a}, \mathfrak{i}, \mathfrak{u}\} \leq \kappa \leq \mathfrak{p}$ and $\lambda \leq \chi Nt(\omega^*)$. Our strategy is to interleave an iteration of length $\lambda\kappa$ and three iterations of length κ . At every stage below $\lambda\kappa$, add another piece of what will be an ultrafilter base that, ordered by \supseteq^* , will be isomorphic to a cofinal subset of $\kappa \times \lambda$. Also, at every stage we will add a pseudointersection, such that the final model satisfies $\mathfrak{p} \geq \kappa$. After each limit stage of cofinality λ , add an element to each of three objects that, when completed, will be a maximal almost disjoint family of size κ , a maximal independent family of size κ , and a base of a simple P_κ -point in ω^* .

Let $\varphi: \lambda^2 \rightarrow \lambda$ be a bijection such that $\varphi(\alpha, \beta) \geq \alpha$ for all $\alpha, \beta < \lambda$. For each $\langle \alpha, \beta \rangle \in \kappa \times \lambda$, set $E_{\alpha, \beta} = \{\langle \gamma, \delta \rangle \in \kappa \times \lambda : \lambda\gamma + \delta < \lambda\alpha + \beta\}$. Suppose $\langle \alpha, \beta \rangle \in \kappa \times \lambda$ and we have constructed $\langle \mathbb{P}_\gamma \rangle_{\gamma \leq \lambda\alpha + \beta}$ to have property (K) and size at most λ at all of its stages, and a sequence $\langle x_{\gamma, \delta} \rangle_{\langle \gamma, \delta \rangle \in E_{\alpha, \beta}}$ of $\mathbb{P}_{\lambda\alpha + \beta}$ -names each forced to be in $[\omega]^\omega$. Set $B = \{x_{\gamma, \delta} : \langle \gamma, \delta \rangle \in E_{\alpha, \beta}\}$. Let $\langle S_\gamma \rangle_{\gamma < \kappa}$ be a partition of λ into κ -many stationary sets such that S_0 contains all successor ordinals. Suppose we have constructed a sequence $\langle \rho_{\gamma, \delta} \rangle_{\langle \gamma, \delta \rangle \in E_{\alpha, \beta}} \in \lambda^{E_{\alpha, \beta}}$ such that we always have $\rho_{\gamma, \delta} \in S_\gamma$ and $\rho_{\gamma, \delta_0} < \rho_{\gamma, \delta_1}$ whenever $\delta_0 < \delta_1$. Set $D_{\alpha, \beta} = \{\langle \gamma, \rho_{\gamma, \delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha, \beta}\}$. Further suppose that $\{\langle \langle \gamma, \rho_{\gamma, \delta} \rangle, x_{\gamma, \delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha, \beta}\}$ is forced to be an order embedding of $D_{\alpha, \beta}$ into $([\omega]^\omega, \supseteq^*)$ and that its range B is forced to have the SFIP. Also suppose that we have the following if $\alpha > 0$.

$$(5.1) \quad \Vdash_{\lambda\alpha + \beta} \forall \sigma \in [B]^{<\omega} \exists \delta < \lambda \bigcap \sigma \not\subseteq^* x_{0, \delta}$$

For each $\varepsilon < \lambda$, set $A_\varepsilon = \{x_{\gamma, \delta} : \langle \gamma, \delta \rangle \in E_{\alpha, \beta} \text{ and } \langle \gamma, \rho_{\gamma, \delta} \rangle < \langle \alpha, \varepsilon \rangle\}$.

Let y_β be a $\mathbb{P}_{\lambda\alpha + \beta}$ -name for a surjection from λ to $[\omega]^\omega$. We may assume that corresponding y_γ have already been constructed for all $\gamma < \beta$. Let $\varphi(\zeta, \eta) = \beta$.

Claim. *If $\alpha > 0$, then we may choose $z \in \{y_\zeta(\eta), \omega \setminus y_\zeta(\eta)\}$ such that*

$$\Vdash_{\lambda_{\alpha+\beta}} \forall \sigma \in [B]^{<\omega} \exists \delta < \lambda \ z \cap \bigcap \sigma \not\subseteq^* x_{0,\delta}.$$

Proof. Suppose not. Let $\{z_0, z_1\} = \{y_\zeta(\eta), \omega \setminus y_\zeta(\eta)\}$. Then, working in a generic extension by $\mathbb{P}_{\lambda_{\alpha+\beta}}$, there exist $\sigma_0, \sigma_1 \in [B]^{<\omega}$ and such that $z_i \cap \bigcap \sigma_i \subseteq^* x_{0,\delta}$ for all $i < 2$ and $\delta < \lambda$. Hence, $\bigcap \bigcup_{i < 2} \sigma_i \subseteq^* x_{0,\delta}$ for all $\delta < \lambda$, in contradiction with (5.1). \square

If $\alpha > 0$, then choose z as in the above claim; otherwise, choose z arbitrarily. If $\alpha = 0$, then set $\rho_{\alpha,\beta} = \beta + 1$. Otherwise, we may choose $\rho_{\alpha,\beta} \in S_\alpha$ such that $\rho_{\alpha,\beta} > \rho_{\alpha,\gamma}$ for all $\gamma < \beta$ and

$$\Vdash_{\lambda_{\alpha+\beta}} \forall \sigma \in [A_{\rho_{\alpha,\beta}}]^{<\omega} \exists \delta < \rho_{\alpha,\beta} \ z \cap \bigcap \sigma \not\subseteq^* x_{0,\delta}.$$

Set $D_{\alpha,\beta+1} = D_{\alpha,\beta} \cup \{\langle \alpha, \rho_{\alpha,\beta} \rangle\}$. Let A' be a $\mathbb{P}_{\lambda_{\alpha+\beta}}$ -name forced to satisfy $A' = A_{\rho_{\alpha,\beta}} \cup \{z\}$ if z splits B and $A' = A_{\rho_{\alpha,\beta}}$ otherwise. Let \mathbb{Q}_0 be a name for the Booth forcing for $A' \cup \{\omega \setminus n : n < \omega\}$; let $x_{\alpha,\beta}$ be a name for a generic pseudointersection of $A' \cup \{\omega \setminus n : n < \omega\}$. (The purpose of $\{\omega \setminus n : n < \omega\}$ is to ensure that $x_{\alpha,\beta}$ does not almost contain any element of $[\omega]^\omega \cap V^{\mathbb{P}_{\lambda_{\alpha+\beta}}}$.)

Let $F_{\lambda_{\alpha+\beta}}$ to be a $\mathbb{P}_{\lambda_{\alpha+\beta}}$ -name for a surjection from λ to the elements of $[[\omega]^\omega]^{<\kappa}$ that have the SFIP. We may assume that corresponding F_γ have already been constructed for all $\gamma < \lambda_{\alpha+\beta}$. Let \mathbb{Q}_1 be a name for the Booth forcing for $F_{\lambda_{\alpha+\zeta}(\eta)}$.

Further suppose we have constructed sequences $\langle w_\gamma \rangle_{\gamma < \alpha}$ and $\langle U_\gamma \rangle_{\gamma < \alpha}$ of $\mathbb{P}_{\lambda_\alpha}$ -names such that $\Vdash_{\lambda_\gamma} U_\delta \cup \{w_\delta\} \subseteq U_\gamma \in \omega^*$ for all $\delta < \gamma < \alpha$, and such that w_γ is forced to be a pseudointersection of U_γ for all $\gamma < \alpha$. If $\beta \neq 0$, then let \mathbb{Q}_2 be a name for the trivial forcing. If $\beta = 0$, then choose U_α such that $\Vdash_{\lambda_\alpha} \bigcup_{\gamma < \alpha} U_\gamma \cup \{w_\gamma\} \subseteq U_\alpha \in \omega^*$, let \mathbb{Q}_2 be a name for the Booth forcing for U_α , and let w_α be a name for a generic pseudointersection of U_α .

Further suppose we have constructed a sequence $\langle a_\gamma \rangle_{\gamma < \alpha}$ of $\mathbb{P}_{\lambda_\alpha}$ -names whose range is forced to be an almost disjoint subfamily of $[\omega]^\omega$. If $\beta \neq 0$, then let \mathbb{Q}_3 be a name for the trivial forcing. If $\beta = 0$, then let \mathbb{Q}_3 be a name for the Booth forcing for $\{\omega \setminus a_\gamma : \gamma < \alpha\}$, and let a_α be a name for a generic pseudointersection of $\{\omega \setminus a_\gamma : \gamma < \alpha\}$.

Further suppose we have constructed a sequence $\langle i_\gamma \rangle_{\gamma < \alpha}$ of $\mathbb{P}_{\lambda_\alpha}$ -names whose range is forced to be an independent subfamily of $[\omega]^\omega$. If $\beta \neq 0$, then let \mathbb{Q}_4 be a name for the trivial forcing. If $\beta = 0$, then set $I = \{i_\gamma : \gamma < \alpha\}$ and let J and x be as in Lemma 5.8; let \mathbb{Q}_4 be a name for the Booth forcing for J ; let i_α be a name for x .

Set $\mathbb{P}_{\lambda_{\alpha+\beta+1}} = \mathbb{P}_{\lambda_{\alpha+\beta}} * \prod_{n < 5} \mathbb{Q}_n$. We may assume $|\prod_{n < 5} \mathbb{Q}_n| \leq \lambda$; hence, $\mathbb{P}_{\lambda_{\alpha+\beta+1}}$ has property (K) and size at most λ . Also, $B \cup \{x_{\alpha,\beta}\}$ is forced to have the SFIP by \mathbb{Q}_0 -genericity because for every $b \in B$ we have that $\{b\} \cup A'$ is forced to have the SFIP because $\{b\} \cup A' \subseteq B \cup \{z\}$ if z splits B and $\{b\} \cup A' \subseteq B$ otherwise. Let us also show that (5.1) holds if we replace β with $\beta+1$. We may assume $\alpha > 0$. Let $\sigma \in [B]^{<\omega}$. Then there exists $\delta < \lambda$ such that $\Vdash_{\lambda_{\alpha+\beta}} z \cap \bigcap (\sigma \cup \tau) \not\subseteq^* x_{0,\delta}$ for all $\tau \in [A_{\rho_{\alpha,\beta}}]^{<\omega}$; hence, $\{(\bigcap \sigma) \setminus x_{0,\delta}\} \cup A'$ is forced to have the SFIP; hence, $\Vdash_{\lambda_{\alpha+\beta+1}} x_{\alpha,\beta} \cap \bigcap \sigma \not\subseteq^* x_{0,\delta}$ by \mathbb{Q}_0 -genericity. Thus, (5.1) holds as desired.

To complete our inductive construction of $\langle \mathbb{P}_\gamma \rangle_{\gamma \leq \lambda_\kappa}$, it suffices to show that $\{\langle \langle \gamma, \rho_{\gamma,\delta} \rangle, x_{\gamma,\delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha,\beta+1}\}$ is forced to be an order embedding of $D_{\alpha,\beta+1}$ into $([\omega]^\omega, \supseteq^*)$. Suppose $\langle \gamma, \delta \rangle \in E_{\alpha,\beta}$. Then $\langle \alpha, \rho_{\alpha,\beta} \rangle \not\leq \langle \gamma, \rho_{\gamma,\delta} \rangle$ and $\Vdash_{\lambda_{\alpha+\beta+1}}$

$x_{\alpha,\beta} \not\sqsupseteq^* x_{\gamma,\delta}$ by \mathbb{Q}_0 -genericity. If $\langle \gamma, \rho_{\gamma,\delta} \rangle < \langle \alpha, \rho_{\alpha,\beta} \rangle$, then $x_{\gamma,\delta} \in A'$; whence, $\Vdash_{\lambda_{\alpha+\beta+1}} x_{\gamma,\delta} \not\sqsupseteq^* x_{\alpha,\beta}$. Suppose $\langle \gamma, \rho_{\gamma,\delta} \rangle \not< \langle \alpha, \rho_{\alpha,\beta} \rangle$. Then $\rho_{\alpha,\beta} < \rho_{\gamma,\delta}$; hence, $\rho_{\gamma,\delta} \geq \rho_{\alpha,\beta} + 1 = \rho_{0,\rho_{\alpha,\beta}}$; hence, $x_{\gamma,\delta} \subseteq^* x_{0,\rho_{\alpha,\beta}}$. By construction, $A' \cup \{\omega \setminus x_{0,\rho_{\alpha,\beta}}\}$ is forced to have the SFIP; hence, $\Vdash_{\lambda_{\alpha+\beta+1}} x_{\gamma,\delta} \subseteq^* x_{0,\rho_{\alpha,\beta}} \not\sqsupseteq^* x_{\alpha,\beta}$ by \mathbb{Q}_0 -genericity. Thus, $\{\langle \langle \gamma, \rho_{\gamma,\delta} \rangle, x_{\gamma,\delta} \rangle : \langle \gamma, \delta \rangle \in E_{\alpha,\beta+1}\}$ is forced to be an embedding as desired.

Let us show that $V^{\mathbb{P}^{\lambda\kappa}}$ satisfies $\lambda \leq \chi \text{Nt}(\omega^*)$. Let G be a generic filter of $\mathbb{P}^{\lambda\kappa}$ and set $\mathcal{B} = \{(x_{\alpha,\beta})_G^* : \langle \alpha, \beta \rangle \in \kappa \times \lambda\}$. Then \mathcal{B} is a local base at some $p \in (\omega^*)^{V[G]}$ because every element of $([\omega]^\omega)^{V[G]}$ is handled by an appropriate \mathbb{Q}_0 . By Lemma 2.15, \mathcal{B} contains a $\chi \text{Nt}(p, \omega^*)^{\text{op}}$ -like local base $\{(x_{\alpha,\beta})_G^* : \langle \alpha, \beta \rangle \in I\}$ at p for some $I \subseteq \kappa \times \lambda$. Set $J = \{\langle \alpha, \rho_{\alpha,\beta} \rangle : \langle \alpha, \beta \rangle \in I\}$. Then J is cofinal in $\kappa \times \lambda$; hence, by Lemma 5.7, J is not ν -like for any $\nu < \lambda$. Hence, $\chi \text{Nt}(\omega^*)^{V[G]} \geq \lambda$.

Finally, let us show that $V^{\mathbb{P}^{\lambda\kappa}}$ satisfies $\max\{\mathfrak{a}, \mathfrak{i}, \mathfrak{u}\} \leq \kappa \leq \mathfrak{p}$. Working in $V[G]$, notice that $\mathfrak{u} \leq \kappa$ because $\bigcup_{\alpha < \kappa} (U_\alpha)_G \in \omega^*$ and $\{(w_\alpha)_G^* : \alpha < \kappa\}$ is a local base at $\bigcup_{\alpha < \kappa} (U_\alpha)_G$. Moreover, $\{(a_\alpha)_G : \alpha < \kappa\}$ and $\{(i_\alpha)_G : \alpha < \kappa\}$ witness that $\mathfrak{a} \leq \kappa$ and $\mathfrak{i} \leq \kappa$. For $\mathfrak{p} \geq \kappa$, note that every element of $([\omega]^\omega)^{<\kappa}$ with the SFIP is $(F_{\lambda_{\alpha+\zeta}}(\eta))_G$ for some $\alpha < \kappa$ and $\zeta, \eta < \lambda$. By \mathbb{Q}_1 -genericity, a pseudointersection of $(F_{\lambda_{\alpha+\zeta}}(\eta))_G$ is added at stage $\lambda\alpha + \varphi(\zeta, \eta)$. \square

Theorem 5.11. $\pi \chi \text{Nt}(\omega^*) = \omega$.

Proof. Fix $p \in \omega^*$. By a result of Balcar and Vojtáš [3], there exists $\langle y_x \rangle_{x \in p}$ such that $y_x \in [x]^\omega$ for all $x \in p$ and $\{y_x\}_{x \in p}$ is an almost disjoint family. Clearly, $\{y_x^*\}_{x \in p}$ is a pairwise disjoint—and therefore ω^{op} -like—local π -base at p . \square

6. POWERS OF ω^*

Definition 6.1. A *box* is a subset E of a product space $\prod_{i \in I} X_i$ such that there exist $\sigma \in [I]^{<\omega}$ and $\langle E_i \rangle_{i \in \sigma}$ such that $E = \bigcap_{i \in \sigma} \pi_i^{-1} E_i$. Let $\text{Nt}_{\text{box}}(\prod_{i \in I} X_i)$ denote the least infinite κ such that $\prod_{i \in I} X_i$ has a κ^{op} -like base of open boxes.

Lemma 6.2 (Peregudov [16]). *In any product space $X = \prod_{i \in I} X_i$, we have $\text{Nt}(X) \leq \text{Nt}_{\text{box}}(X) \leq \sup_{i \in I} \text{Nt}(X_i)$.*

Lemma 6.3 (Malykhin [15]). *Let $X = \prod_{i \in I} X_i$ where each X_i is a nonsingleton T_1 space. If $w(X) \leq |I|$, then $\text{Nt}(X) = \text{Nt}_{\text{box}}(X) = \omega$.*

Remark. In Lemma 6.3, the hypothesis that the factor spaces be nonsingleton and T_1 can be weakened to merely require that each factor space is the union of two nontrivial open sets. Also, the conclusion of Lemma 6.3 may be amended with the statement that X has a $\langle |I|, \omega \rangle$ -splitter: use $\{\{\pi_i^{-1} U_i, \pi_i^{-1} V_i\}\}_{i \in I}$ where each $\{U_i, V_i\}$ is a nontrivial open cover of X_i .

Theorem 6.4. *The sequence $\langle \text{Nt}((\omega^*)^{\omega+\alpha}) \rangle_{\alpha \in \mathbb{O}_n}$ is nonincreasing. Moreover, $\text{Nt}((\omega^*)^\mathfrak{c}) = \omega$.*

Proof. Note that if $\omega \leq \alpha \leq \beta$, then $(\omega^*)^\beta \cong ((\omega^*)^\alpha)^\beta$. Then apply Lemmas 6.2 and 6.3. \square

Lemma 6.5. *Let $0 < n < \omega$ and X be a space. Then $\text{Nt}_{\text{box}}(X^n) = \text{Nt}(X)$.*

Proof. Set $\kappa = \text{Nt}_{\text{box}}(X^n)$. By Lemma 6.2, $\kappa \leq \text{Nt}(X)$. Let us show that $\text{Nt}(X) \leq \kappa$. Let \mathcal{A} be a κ^{op} -like base of X^n consisting only of boxes. Let \mathcal{B} denote the set of all nonempty open $V \subseteq X$ for which there exists $\prod_{i < n} U_i \in \mathcal{A}$ such that $V = \bigcap_{i < n} U_i$. Then \mathcal{B} is a base of X because if $p \in U$ and U is an open subset of X , then there

exists $\prod_{i < n} U_i \in \mathcal{A}$ such that $\langle p \rangle_{i < n} \in \prod_{i < n} U_i \subseteq U^n$; whence, $p \in \bigcap_{i < n} U_i \subseteq U$ and $\bigcap_{i < n} U_i \in \mathcal{B}$.

It suffices to show that \mathcal{B} is κ^{op} -like. Suppose not. Then there exist $\prod_{i < n} U_i \in \mathcal{A}$ and $\langle \prod_{i < n} V_{\alpha, i} \rangle_{\alpha < \kappa} \in \mathcal{A}^\kappa$ such that

$$\emptyset \neq \bigcap_{i < n} U_i \subseteq \bigcap_{i < n} V_{\alpha, i} \neq \bigcap_{i < n} V_{\beta, i}$$

for all $\alpha < \beta < \kappa$. Clearly, $\prod_{i < n} V_{\alpha, i} \neq \prod_{i < n} V_{\beta, i}$ for all $\alpha < \beta < \kappa$. Choose $U \in \mathcal{A}$ such that $U \subseteq (\bigcap_{i < n} U_i)^n$. Then $U \subseteq \prod_{i < n} V_{\alpha, i}$ for all $\alpha < \kappa$, in contradiction with how we chose \mathcal{A} . \square

Lemma 6.6. *If $0 < n < \omega$ and X is a compact space such that $\chi(p, X) = w(X)$ for all $p \in X$, then $Nt(X) = Nt(X^n)$.*

Proof. By Lemma 6.5, it suffices to show that $Nt_{\text{box}}(X^n) \leq Nt(X^n)$. By Lemma 2.7, either X^n has a $\langle w(X^n), Nt(X^n) \rangle$ -splitter, or $Nt(X^n) = w(X^n)^+$. Hence, by Lemma 2.6, $Nt_{\text{box}}(X^n) \leq Nt(X^n)$. \square

Theorem 6.7. *If $0 < n < \omega$, then $Nt(\omega^*) \geq Nt((\omega^*)^n) \geq \min\{Nt(\omega^*), \mathfrak{c}\}$. Moreover, $\max\{\mathfrak{u}, \text{cf } \mathfrak{c}\} = \mathfrak{c}$ implies $Nt(\omega^*) = Nt((\omega^*)^n)$.*

Proof. Lemma 6.2 implies $Nt(\omega^*) \geq Nt((\omega^*)^n)$. To prove the rest of the theorem, first consider the case $\mathfrak{r} < \mathfrak{c}$. As in the proof of Theorem 2.3, construct a point $p \in \omega^*$ such that $\pi\chi(p, \omega^*) = \mathfrak{r}$ and $\chi(p, \omega^*) = \mathfrak{c}$. Then $\pi\chi(\langle p \rangle_{i < n}, (\omega^*)^n) = \mathfrak{r}$ and $\chi(\langle p \rangle_{i < n}, (\omega^*)^n) = \mathfrak{c}$; hence, $Nt((\omega^*)^n) \geq \mathfrak{c}$ by Theorem 2.1. Moreover, if $\text{cf } \mathfrak{c} = \mathfrak{c}$, then $Nt((\omega^*)^n) = Nt(\omega^*) = \mathfrak{c}^+$. If $\mathfrak{u} = \mathfrak{c}$, then $Nt(\omega^*) = Nt((\omega^*)^n)$ by Lemma 6.6. Finally, in the case $\mathfrak{r} = \mathfrak{c}$, we have $\mathfrak{u} = \mathfrak{c}$, which again implies $Nt(\omega^*) = Nt((\omega^*)^n)$. \square

Corollary 6.8. *Suppose $\max\{\mathfrak{u}, \text{cf } \mathfrak{c}\} = \mathfrak{c}$. Then $\langle Nt((\omega^*)^{1+\alpha}) \rangle_{\alpha \in \text{On}}$ is nonincreasing.*

Proof. By Theorem 6.7 and Lemma 6.2, $Nt((\omega^*)^n) = Nt(\omega^*) \geq Nt((\omega^*)^\alpha)$ whenever $0 < n < \omega \leq \alpha$. The rest follows from Theorem 6.4. \square

Theorem 6.9. *Suppose $\mathfrak{u} = \mathfrak{c}$. Then $Nt((\omega^*)^{1+\alpha}) = Nt(\omega^*)$ for all $\alpha < \text{cf } \mathfrak{c}$.*

Proof. Let λ be an arbitrary infinite cardinal less than $Nt(\omega^*)$. By Lemma 2.7, it suffices to show that $(\omega^*)^{1+\alpha}$ does not have a $\langle \mathfrak{c}, \lambda \rangle$ -splitter. Seeking a contradiction, suppose $\langle \mathcal{F}_\beta \rangle_{\beta < \mathfrak{c}}$ is such a $\langle \mathfrak{c}, \lambda \rangle$ -splitter. We may assume $\bigcup_{\beta < \mathfrak{c}} \mathcal{F}_\beta$ consists only of open boxes because we can replace each \mathcal{F}_β with a suitable refinement. Since $\alpha < \text{cf } \mathfrak{c}$, there exist $\sigma \in [1 + \alpha]^{< \omega}$ and $I \in [\mathfrak{c}]^\mathfrak{c}$ such that, for every $U \in \bigcup_{\beta \in I} \mathcal{F}_\beta$, there exists $\varphi(U) \subseteq (\omega^*)^\sigma$ such that $U = \pi_\sigma^{-1} \varphi(U)$. Let j be a bijection from \mathfrak{c} to I . Then $\langle \varphi^{\mathcal{F}_{j(\beta)}} \rangle_{\beta < \mathfrak{c}}$ is a $\langle \mathfrak{c}, \lambda \rangle$ -splitter of $(\omega^*)^\sigma$. Hence, $Nt((\omega^*)^\sigma) \leq \lambda < Nt(\omega^*)$ by Lemma 2.6. But $Nt((\omega^*)^\sigma) < Nt(\omega^*)$ contradicts Theorem 6.7. \square

Lemma 6.10. *Suppose a space X has a $\langle \text{cf } w(X), \text{cf } w(X) \rangle$ -splitter. Then $Nt(X) \leq w(X)$.*

Proof. Set $\kappa = \text{cf } w(X)$ and $\lambda = w(X)$. Let $\langle \mathcal{F}_\alpha \rangle_{\alpha < \kappa}$ be a $\langle \kappa, \kappa \rangle$ -splitter of X . Let $h : \lambda \rightarrow \kappa$ satisfy $|h^{-1}\{\alpha\}| < \lambda$ for all $\alpha < \kappa$. Then $\langle \mathcal{F}_{h(\alpha)} \rangle_{\alpha < \lambda}$ is a $\langle \lambda, \lambda \rangle$ -splitter because if $I \in [\lambda]^\lambda$, then $h^{\mathcal{I}} \in [\kappa]^\kappa$. By Lemma 2.6, $Nt(X) \leq \lambda$. \square

Remark. The proof of the above lemma shows that for any infinite cardinal κ , a space with a $\langle \text{cf } \kappa, \text{cf } \kappa \rangle$ -splitter also has a $\langle \kappa, \kappa \rangle$ -splitter.

Theorem 6.11. $Nt((\omega^*)^{\text{cf } \mathfrak{c}}) \leq \mathfrak{c}$.

Proof. The sequence $\langle \{\pi_\alpha^{-1}(\{2n : n < \omega\}^*), \pi_\alpha^{-1}(\{2n+1 : n < \omega\}^*)\} \rangle_{\alpha < \text{cf } \mathfrak{c}}$ is a $\langle \text{cf } \mathfrak{c}, \omega \rangle$ -splitter of $(\omega^*)^{\text{cf } \mathfrak{c}}$. Apply Lemma 6.10. \square

Theorem 6.12. For all cardinals κ satisfying $\kappa > \text{cf } \kappa > \omega_1$, it is consistent that $\mathfrak{c} = \kappa$ and $\mathfrak{r} < \text{cf } \mathfrak{c}$. The last inequality implies $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}^+$ for all $\alpha < \text{cf } \mathfrak{c}$ and $Nt((\omega^*)^\beta) = \mathfrak{c} = \kappa$ for all $\beta \in \mathfrak{c} \setminus \text{cf } \mathfrak{c}$.

Proof. Starting with $\mathfrak{c} = \kappa$ in the ground model, the proof of Theorem 2.3 shows how to force $\mathfrak{r} = \mathfrak{u} = \omega_1$ while preserving \mathfrak{c} . Now suppose $\mathfrak{r} < \text{cf } \mathfrak{c}$. Fix $\alpha < \text{cf } \mathfrak{c}$ and $\beta \in \mathfrak{c} \setminus \text{cf } \mathfrak{c}$. By Theorems 6.11 and 6.4, $Nt((\omega^*)^\beta) \leq \mathfrak{c}$. To see that $Nt((\omega^*)^\beta) \geq \mathfrak{c}$, proceed as in the proof of Theorem 6.7, constructing a point with character \mathfrak{c} and π -character $|\beta|$. Similarly prove $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}^+$ by constructing a point with character \mathfrak{c} and π -character $|\mathfrak{r} + \alpha|$. \square

Lemma 6.13. Suppose κ , λ , and μ are cardinals and p is a point in a product space $X = \prod_{\alpha < \kappa} X_\alpha$ satisfying the following for all $\alpha < \kappa$.

- (1) $0 < \kappa < w(X)$ and $\omega \leq \lambda \leq w(X)$.
- (2) $\kappa < \text{cf } w(X)$ or $\lambda < w(X)$.
- (3) $\mu < \lambda$ or $\mu = \text{cf } \lambda$.
- (4) $\chi(p(\alpha), X_\alpha) < \lambda$ or the intersection of any μ -many neighborhoods of $p(\alpha)$ has nonempty interior.

Then $\chi(p, X) < w(X)$ or $Nt(X) > \mu$.

Proof. Let \mathcal{A} be a base of X . Set $\mathcal{B} = \{U \in \mathcal{A} : p \in U\}$. For each $\alpha < \kappa$, let \mathcal{C}_α be a local base at $p(\alpha)$ of size $\chi(p(\alpha), X_\alpha)$. Set $F = \bigcup_{r \in [\kappa]^{< \omega}} \prod_{\alpha \in r} \mathcal{C}_\alpha$. For each $\sigma \in F$, set $U_\sigma = \bigcap_{\alpha \in \text{dom } \sigma} \pi_\alpha^{-1} \sigma(\alpha)$. For each $V \in \mathcal{B}$, choose $\sigma(V) \in F$ such that $p \in U_{\sigma(V)} \subseteq V$. We may assume $\chi(p, X) = w(X)$; hence, by (1) and (2), there exist $r \in [\kappa]^{< \omega}$ and $\mathcal{D} \in [\mathcal{B}]^\lambda$ such that $\text{dom } \sigma(V) = r$ for all $V \in \mathcal{D}$. Set $s = \{\alpha \in r : \chi(p(\alpha), X_\alpha) < \lambda\}$ and $t = r \setminus s$. By (3), there exist $\tau \in \prod_{\alpha \in s} \mathcal{C}_\alpha$ and $\mathcal{E} \in [\mathcal{D}]^\mu$ such that $\sigma(V) \upharpoonright s = \tau$ for all $V \in \mathcal{E}$. By (4), $\bigcap_{V \in \mathcal{E}} \sigma(V)(\alpha)$ has nonempty interior for all $\alpha \in t$. Hence, $\bigcap \mathcal{E}$ has nonempty interior because it contains $U_\tau \cap \bigcap_{\alpha \in t} \pi_\alpha^{-1} \bigcap_{V \in \mathcal{E}} \sigma(V)(\alpha)$. Thus, $Nt(X) > \mu$. \square

Theorem 6.14. Suppose $0 < \alpha < \mathfrak{c}$ and $\langle X_\beta \rangle_{\beta < \alpha}$ is a sequence of spaces each with weight at most \mathfrak{c} . Set $X = \prod_{\beta < \alpha} (X_\beta \oplus \omega^*)$. Then $Nt(X) \geq \mathfrak{p}$.

Proof. Let ν be an arbitrary infinite cardinal less than \mathfrak{p} . Set $\kappa = |\alpha|$, $\lambda = \nu^+$, and $\mu = \nu$. Choose $q \in \omega^*$ such that $\chi(q, \omega^*) = \mathfrak{c}$; set $p = \langle q \rangle_{\beta < \alpha}$. Then Lemma 6.13 applies because if $\kappa \geq \text{cf } w(X) = \text{cf } \mathfrak{c}$, then $\lambda \leq \mathfrak{p} \leq \text{cf } \mathfrak{c} < \mathfrak{c} = w(X)$. Therefore, $Nt(X) > \nu$. \square

Corollary 6.15. Suppose $\mathfrak{p} = \mathfrak{c}$. Then $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}$ for all $\alpha < \mathfrak{c}$.

Proof. By Theorem 2.11, $Nt(\omega^*) \leq \mathfrak{c}$. Hence, by Corollary 6.8, $Nt((\omega^*)^{1+\alpha}) \leq \mathfrak{c}$ for all $\alpha \in \text{On}$. By Theorem 6.14, $Nt((\omega^*)^{1+\alpha}) = \mathfrak{c}$ for all $\alpha < \mathfrak{c}$. \square

Corollary 6.16. Suppose $\alpha < \mathfrak{c}$ and $\langle X_\beta \rangle_{\beta < \alpha}$ is a sequence of spaces each with weight at most \mathfrak{c} . Then $\prod_{\beta < \alpha} (X_\beta \oplus \omega^*)$ is not homeomorphic to a product of \mathfrak{c} -many nonsingleton spaces.

Proof. Combine Theorem 6.14 and Lemma 6.3. □

7. QUESTIONS

Question 1. Is it consistent that $Nt(\omega^*) = \mathfrak{c}^+$ and $\mathfrak{r} \geq \text{cf } \mathfrak{c}$?

Question 2. Is $Nt(\omega^*) < \mathfrak{ss}_\omega$ consistent? This inequality implies $\mathfrak{u} < \mathfrak{c}$. Hence, by Theorem 2.11, the inequality further implies

$$\text{cf } \mathfrak{c} \leq \mathfrak{r} \leq \mathfrak{u} < \mathfrak{c} = Nt(\omega^*) < \mathfrak{ss}_\omega = \mathfrak{c}^+.$$

More generally, does any space X have a base that does not contain an $Nt(X)^{\text{op}}$ -like base?

Question 3. Is $\mathfrak{ss}_\omega < \mathfrak{ss}_2$ consistent?

Question 4. Letting \mathfrak{g} denote the groupwise density number (see 6.26 of [7]), is $Nt(\omega^*) < \mathfrak{g}$ consistent? $\chi Nt(\omega^*) < \mathfrak{g}$? In particular, what are $Nt(\omega^*)$ and $\chi Nt(\omega^*)$ in the Laver model (see 11.7 of [7])?

Question 5. Is $\text{cf } Nt(\omega^*) < Nt(\omega^*) < \mathfrak{c}$ consistent? $\text{cf } Nt(\omega^*) = \omega$?

Question 6. Is $\text{cf } \mathfrak{c} < Nt(\omega^*) < \mathfrak{c}$ consistent?

Question 7. What is $\chi Nt(\omega^*)$ in the forcing extension of the proof of Theorem 4.13? More generally, is it consistent that $\chi Nt(\omega^*) < Nt(\omega^*) \leq \mathfrak{c}$?

Question 8. Is $\chi Nt(\omega^*) = \omega$ consistent? An affirmative answer would be a strengthening of Shelah's result [19] that ω^* consistently has no P-points. If the answer is negative, then which, if any, of \mathfrak{p} , \mathfrak{h} , \mathfrak{s} , and \mathfrak{g} are lower bounds of $\chi Nt(\omega^*)$ in ZFC?

Question 9. Is $\text{cf } \mathfrak{c} < \chi Nt(\omega^*)$ consistent? $\text{cf } \mathfrak{c} < \chi Nt(\omega^*) < \mathfrak{c}$?

Question 10. Does any Hausdorff space have uncountable local Noetherian π -type? (It is easy to construct such T_1 spaces: give $\omega_1 + 1$ the topology $\{(\omega_1 + 1) \setminus (\alpha \cup \sigma) : \alpha < \omega_1 \text{ and } \sigma \in [\omega_1 + 1]^{<\omega}\} \cup \{\emptyset\}$.)

Question 11. Is it consistent that $Nt((\omega^*)^{1+\alpha}) < \min\{Nt(\omega^*), \mathfrak{c}\}$ for some $\alpha < \mathfrak{c}$? Is it consistent that $Nt((\omega^*)^{1+\alpha}) < Nt(\omega^*)$ for some $\alpha < \text{cf } \mathfrak{c}$?

REFERENCES

- [1] B. Balcar, J. Pelant, and P. Simon, *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math. **110** (1980), no. 1, 11–24.
- [2] B. Balcar and P. Simon, *Reaping number and π -character of Boolean algebras*, Topological, algebraical and combinatorial structures. Discrete Math. **108** (1992), no. 1-3, 5–12.
- [3] B. Balcar and P. Vojtáš, *Almost disjoint refinement of families of subsets of N* , Proc. Amer. Math. Soc., **79**, no. 3, 1980.
- [4] T. Bartoszyński and H. Judah, *Set theory. On the structure of the real line*, A K Peters, Ltd., Wellesley, MA, 1995.
- [5] J. E. Baumgartner and P. Dordal, *Adjoining Dominating Functions*, J. Symbolic Logic **50** (1985), no. 1, 94–101.
- [6] M. G. Bell, *On the combinatorial principle $P(\mathfrak{c})$* , Fund. Math. **114** (1981), no. 2, 149–157.
- [7] A. Blass, *Combinatorial Cardinal Characteristics of the Continuum*. In M. Foreman, A. Kanamori, and M. Magidor, eds., *Handbook of Set Theory*. Kluwer, to appear.
- [8] A. Dow and J. Zhou, *Two real ultrafilters on ω* , Topology Appl. **97** (1999), no. 1-2, 149–154.
- [9] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 2nd ed., 1989.
- [10] M. Groszek and T. Jech, *Generalized iteration of forcing*, Trans. Amer. Math. Soc. **324** (1991), no. 1, 1–26.
- [11] J. Isbell, *The category of cofinal types. II*, Trans. Amer. Math. Soc. **116** (1965), 394–416.

- [12] I. Juhász, *Cardinal functions in topology—ten years later*, Mathematical Centre Tracts, 123, Mathematisch Centrum, Amsterdam, 1980.
- [13] K. Kunen, *Random and Cohen reals*, Handbook of set-theoretic topology, 887–911, North-Holland, Amsterdam, 1984.
- [14] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1980.
- [15] V. I. Malykhin, *On Noetherian Spaces*, Amer. Math. Soc. Transl. **134** (1987), no. 2, 83–91.
- [16] S. A. Peregudov, *On the Noetherian type of topological spaces*, Comment. Math. Univ. Carolin. **38** (1997), no. 3, 581–586.
- [17] B. Pospíšil, *On bicomact spaces*, Publ. Fac. Sci. Univ. Masaryk (1939), no. 270.
- [18] J. Roitman, *Correction to: Adding a random or a Cohen real: topological consequences and the effect on Martin's axiom*, Fund. Math. **129** (1988), no. 2, 141.
- [19] S. Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982.
- [20] S. Shelah, *Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing*, Acta Math. **192** (2004), no. 2, 187–223

UNIVERSITY OF WISCONSIN-MADISON MATHEMATICS DEPT.
E-mail address: milovich@math.wisc.edu