

# FORBIDDEN RECTANGLES IN COMPACTA

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ABSTRACT. We establish negative results about “rectangular” local bases in compacta. For example, there is no compactum where all points have local bases of cofinal type  $\omega \times \omega_2$ . For another, the compactum  $\beta\omega$  has no nontrivially rectangular local bases, and the same is consistently true of  $\beta\omega \setminus \omega$ : no local base in  $\beta\omega$  has cofinal type  $\kappa \times \mathfrak{c}$  if  $\kappa < \mathfrak{m}_{\sigma-n\text{-linked}}$  for some  $n \in [1, \omega)$ . Also, CH implies that every local base in  $\beta\omega \setminus \omega$  has the same cofinal type as one in  $\beta\omega$ .

We also answer a question of Dobrinen and Todorćević about cofinal types of ultrafilters: the Fubini square of a filter on  $\omega$  always has the same cofinal type as its Fubini cube. Moreover, the Fubini product of nonprincipal P-filters on  $\omega$  is commutative modulo cofinal equivalence.

## 1. INTRODUCTION

Recall that a space  $X$  is *homogeneous* if for all  $p, q \in X$ ,  $h(p) = q$  for some autohomeomorphism of  $X$ . Many questions about compact homogeneous spaces are still unsolved (in all models of ZFC) after decades; see [13] for a survey of these questions. For example, Rudin’s Problem asks whether every homogeneous compactum has a convergent sequence. (A *compactum* is a compact Hausdorff space.) Our motivating question is Van Douwen’s Problem: is there a homogeneous compactum  $X$  with a pairwise disjoint family  $\mathcal{F}$  of open sets such that  $|\mathcal{F}| > |\mathbb{R}|$ . See [11] for more about Van Douwen’s Problem.

Pairwise disjoint families of open sets are called *cellular families* for short; the *cellularity*  $c(X)$  of a space  $X$  is the supremum of the cardinalities of its cellular families. Oversimplifying, Van Douwen’s Problem is hard because if we seek an infinitary operation on spaces that preserves both homogeneity and compactness, we apparently find only the product operation (and special quotients of products [15], but quotients cannot increase cellularity). However, the cellularity of a product is just the supremum of the cellularities of its finite subproducts.

What might a homogeneous compactum with large cellularity “look like”? One way to make this question more precise is to ask for examples of directed sets  $(D, \leq)$  such that any homogeneous compactum  $X$  with a local base  $\mathcal{B}$  satisfying  $(\mathcal{B}, \supseteq) \cong (D, \leq)$  will satisfy some lower bound of  $c(X)$ . Since every infinite compactum has a countable set with a limit point, homogeneity implies that any such  $D$  must have a countable unbounded set. However, for any directed  $D$ ,  $\omega \times D$  has a countable unbounded set. So, for a simple, “rectangular” example, if any space  $X$  has a clopen local base  $\mathcal{B}$  of order type  $\omega \times \kappa$  (where  $\kappa$  is an infinite cardinal), then  $c(X) \geq \kappa$  because if  $U: \omega \times \kappa \cong \mathcal{B}$ , then  $\{U(0, i) \setminus U(0, i + 1) : i < \kappa\}$  is a cellular family. In

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Section 2, we will remove the assumption of clopenness in exchange for assuming merely that  $X$  is  $T_3$ . Moreover, we will relax “order type” to “cofinal type.”

Two local bases at a common point  $p$  in a space  $X$  could have different order types, but with respect to the containment ordering  $\supseteq$ , they are both cofinal subsets of the neighborhood filter  $\text{Nbhd}(p, X)$  (our notation for the set of all  $U \subseteq X$  where  $p$  is in the interior of  $U$ ). Therefore, it is more natural to investigate a local base’s cofinal type than its order type, where preorders  $P$  and  $Q$  are *cofinally equivalent* if there is a preorder  $R$  such that  $P$  and  $Q$  are order isomorphic to cofinal subsets of  $R$ . (It is not hard to check that this is an equivalence relation.) Since all local bases at  $p$  are cofinally equivalent to  $\text{Nbhd}(p, X)$ , we will state our results in terms of the cofinal type of  $\text{Nbhd}(p, X)$ , with the convention that families of subsets of a space are ordered by containment.

In section 2, we establish that if a  $T_3$  space  $X$  has a neighborhood filter cofinally equivalent to a finite product  $\prod_{i \leq n} \kappa_i$  of regular cardinals  $\kappa_0 < \dots < \kappa_n$ , then  $c(X) \geq \kappa_n$ . In section 3, we prove that not all points in a compactum can have a fixed “skinny” cofinal type. For example, given  $\kappa_0 < \dots < \kappa_n$  as above, if  $\kappa_m^+ < \kappa_{m+1}$  for some  $m < n$ , then not all neighborhood filters in a compactum can be cofinally equivalent to  $\prod_{i \leq n} \kappa_i$ . So, in a homogeneous compactum, no neighborhood filter is cofinally equivalent to such  $\prod_{i \leq n} \kappa_i$ . Section 3 then goes on to stronger theorem for homogeneous compacta in models of GCH. A corollary is that the supremum of the cardinalities of the free sequences in a homogeneous compactum  $X$  is always attained if GCH holds. Left open is whether a homogeneous compactum could have a neighborhood filter cofinally equivalent to  $\prod_{i \leq n} \omega_i$  for some  $n \geq 1$ .

In section 4, we shift our attention to the inhomogeneous compactum  $\beta\omega \setminus \omega$ . In this space, every point corresponds to a nonprincipal ultrafilter on  $\omega$ , and the cofinal type of a neighborhood filter (ordered by containment) is the same as the cofinal type of the corresponding ultrafilter ordered by eventual containment. Among other things, we observe that  $\beta\omega \setminus \omega$  consistently has no neighborhood filter cofinally equivalent to any  $\prod_{i \leq n} \kappa_i$  as above. In ZFC, it is known that no neighborhood filter of  $\beta\omega \setminus \omega$  is cofinally equivalent to  $\omega \times \mathfrak{c}$  [18]. We extend this result, ruling out  $\kappa \times \mathfrak{c}$  for all  $\kappa < \sup_{n < \omega} \mathfrak{m}_{\sigma-n\text{-linked}}$ , which is at worst very close to optimal because it is consistent to have  $\mathfrak{m}_{\sigma\text{-centered}} < \mathfrak{c}$  and a neighborhood filter of  $\beta\omega \setminus \omega$  cofinally equivalent to  $\mathfrak{m}_{\sigma\text{-centered}} \times \mathfrak{c}$ .

In section 5, we prove some results about cofinal types of neighborhood bases in  $\beta\omega$ , which are exactly the cofinal types of neighborhood bases of ultrafilters on  $\omega$ . In particular, we answer a question of Dobrinen and Todorćević [7] by showing that the Fubini square and Fubini cube of a filter on  $\omega$  are always cofinally equivalent. (The ordering is containment.) We also establish commutativity modulo cofinal equivalence for the Fubini product of nonprincipal  $P$ -filters on  $\omega$ . These results follow from our “rectangular” characterization of the Fubini product: if  $F$  and  $G$  are nonprincipal filters on  $\omega$ , then the Fubini product of  $F$  and  $G$  is cofinally equivalent to  $F \times G^\omega$ .

## 2. RECTANGLES AND CELLULARITY

Given an ordinal  $\alpha$ , let  $2_{\text{lex}}^\alpha$  be  ${}^\alpha 2$  with the topology induced by the lexicographic ordering. If  $\kappa_0 < \dots < \kappa_n$  are regular ordinals and  $\lambda_i$  is a regular ordinal  $\leq \kappa_i$ , for all  $i$ , then it easy to find a point  $p$  in  $\prod_{i \leq n} 2_{\text{lex}}^{\kappa_i}$  such that each  $p(i)$  has cofinality

$\lambda_i$  and coinitiality  $\kappa_i$  in  $2_{\text{lex}}^{\kappa_i}$ , where our convention is that 1 is the unique finite regular ordinal, every minimum of a preorder has cofinality 1, and every maximum of a preorder has coinitiality 1. Conversely, for every  $p \in \prod_{i \leq n} 2_{\text{lex}}^{\kappa_i}$  and  $i \leq n$ , one of  $\text{cf}(p(i))$  and  $\text{ci}(p(i))$  is  $\kappa_i$ , and the other is some regular  $\lambda_i \leq \kappa_i$ . Therefore, the spectrum of cofinal types of  $\prod_{i \leq n} 2_{\text{lex}}^{\kappa_i}$  is  $\left\{ \prod_{i \leq n} (\lambda_i \times \kappa_i) / \equiv_{\text{cf}} : \kappa_i \geq \lambda_i \in \text{Reg} \right\}$  where Reg is the class of regular ordinals.

Since the diagonal of  $\kappa_i \times \kappa_i$  is cofinal,  $\kappa_i \times \kappa_i \equiv_{\text{cf}} \kappa_i \equiv_{\text{cf}} 1 \times \kappa_i$ . Therefore, all neighborhood filters of  $\prod_{i \leq n} 2_{\text{lex}}^{\omega_i}$  are cofinally equivalent to  $\prod_{i \leq n} \omega_i$ . However, this space, though compact Hausdorff, is not homogeneous if  $n \geq 1$ . To see this, recall that the  $\pi$ -character  $\pi\chi(p, X)$  of a point  $p$  in a space  $X$  is the minimum of the cardinalities of the *local*  $\pi$ -bases at  $p$ , where a local  $\pi$ -base at  $p$  is a family of nonempty open sets that includes a subset of every neighborhood of  $p$ . If some point  $p$  in a linearly ordered space  $X$  has cofinality  $\lambda \geq \omega$ , then  $p$  has a local  $\pi$ -base of size  $\lambda$ :  $\{(q_i, q_{i+1}) : i < \lambda\}$  for some increasing sequence  $\vec{q}$  converging to  $p$ . Moreover,  $\pi\chi(p, X) = \min(\{\text{cf}(p), \text{ci}(p)\} \setminus \{1\})$ . Therefore, if  $p \in \prod_{i \leq n} 2_{\text{lex}}^{\omega_i}$ , then

$$\pi\chi(p, X) = \max_{i \leq n} \min(\{\text{cf}(p(i)), \text{ci}(p(i))\} \setminus \{1\}).$$

It follows that  $\prod_{i \leq n} 2_{\text{lex}}^{\omega_i}$  has points with  $\pi$ -character  $\omega_i$ , for all  $i \leq n$ . Thus,  $\prod_{i \leq n} 2_{\text{lex}}^{\omega_i}$  is only homogeneous in the trivial case  $n = 0$ . More generally, it is shown in [3] that for any homogeneous compact product of linear orders, all factors  $X$  are such that every  $p \in X$  satisfies  $\{\text{cf}(p, X), \text{ci}(p, X)\} \subseteq \{1, \omega\}$ .

*Question 2.1.* Is there a homogeneous compactum with a neighborhood filter cofinally equivalent to  $\prod_{i \leq n} \omega_i$  for some  $n \geq 1$ ?

There is some weak evidence in [16] for a “no” answer to the above question. Suppose that  $Y$  is a homogeneous compactum with a neighborhood filter cofinally equivalent to  $\prod_{i \leq n} \omega_i$  for some  $n \geq 1$ . If  $Y$  also had a point of uncountable  $\pi$ -character, then, by Theorem 5.7 of [16], there would be a Tukey map (see Definition 4.6) from  $[\omega_1]^{<\omega}$  (ordered by  $\subseteq$ ) to  $\prod_{i \leq n} \omega_i$ . However, it is well known that there is no such Tukey map. (For a quick proof, check that every uncountable subset of  $\prod_{i \leq n} \omega_i$  has an infinite bounded subset, and that this property is precisely the negation of having a Tukey map from  $[\omega_1]^{<\omega}$ .) Thus, a positive answer to Question 2.1 requires a homogeneous compactum whose points all have  $\pi$ -character  $\omega$  but character  $\omega_n$ . (See Definition 3.12.) In almost all known homogeneous compacta  $X$ ,  $\pi$ -character equals character at all points. The only known class of exceptions was discovered by Van Mill in [12], and even these exceptions consistently do not exist: they are homogeneous if  $\text{MA} + \neg\text{CH}$  holds but inhomogeneous if  $\text{CH}$  holds. Moreover, it is shown in [16] that in all known homogeneous compacta  $X$ , all points  $p$  are *flat*, that is, satisfy  $\text{Nbhd}(p, X) \equiv_{\text{cf}} [\kappa]^{<\omega}$  where  $\kappa$  is the character of  $p$ . (In particular, Van Mill’s exceptional homogeneous compacta are all separable and have weight less than  $\mathfrak{p}$ ; by Theorem 2.16 of [16], any homogeneous compactum satisfying these two properties has only flat points.)

Question 2.1 is relevant to Van Douwen’s Problem because of the next theorem.

**Theorem 2.2.** *If  $\kappa_0 < \dots < \kappa_n$  are regular cardinals,  $X$  is  $T_3$  space,  $p \in X$ , and  $\text{Nbhd}(p, X) \equiv_{\text{cf}} \prod_{i \leq n} \kappa_i$ , then  $c(X) \geq \kappa_n$ .*

We will actually prove a stronger result and deduce the above theorem as a corollary.

**Definition 2.3.** Given a point  $p$  in a space  $X$ ,

- an *escape sequence* at  $p$  is a transfinite sequence of neighborhoods  $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_i \supseteq \cdots$  of  $p$  such that  $\bigcap_i U_i$  is not a neighborhood of  $p$ ;
- $\text{Escape}(p, X)$  is the set of infinite cardinals  $\kappa$  for which  $p$  has a  $\kappa$ -long escape sequence;
- $\text{Escape}_{\text{RO}}(p, X)$  is the set of infinite cardinals  $\kappa$  for which  $p$  has a  $\kappa$ -long escape sequence consisting of regular open sets;
- $\hat{c}(X)$  is the least cardinal  $\kappa$  such that  $X$  lacks a cellular family of size  $\kappa$ .  
(For increased precision at limit cardinals, we use  $\hat{c}(X)$  instead of  $c(X)$ .)

**Theorem 2.4.** *If  $X$  is a space and  $\kappa \in \text{Escape}_{\text{RO}}(p, X) \cap \text{Reg}$  for some  $p \in X$ , then  $\hat{c}(X) > \kappa$ .*

*Proof.* Let  $\vec{U}$  be a  $\kappa$ -long regular open escape sequence at  $p$ . Since  $\kappa$  is regular and  $\vec{U}$  cannot be eventually constant, we may thin out  $\vec{U}$  such that it is strictly decreasing. Since each  $U_i$  and  $U_{i+1}$  are regular open, each  $U_i \setminus \bar{U}_{i+1}$  is nonempty, so  $\{U_i \setminus \bar{U}_{i+1} : i < \kappa\}$  is a cellular family of size  $\kappa$ .  $\square$

**Example 2.5.** If  $X = 2^{\omega_1}$  (with the product topology), then there is an  $\omega_1$ -long escape sequence  $\vec{U}$  at  $\vec{0}$  given by  $U_i = \bigcup_{i \leq j < \alpha} \pi_j^{-1}[\{0\}]$ . However,  $\hat{c}(X) = \omega_1$  (by a well-known  $\Delta$ -system argument). So, by Theorem 2.4, at no point in  $X$  is there a regular open escape sequence with length  $\omega_1$ . Thus, despite the set of regular open neighborhoods of  $\vec{0}$  being cofinally equivalent with the set of open neighborhoods of  $\vec{0}$ , the former never has unbounded increasing  $\omega_1$ -sequences, while the latter does. Moreover, the neighborhood filter of  $\vec{0}$  in  $X$  is cofinally equivalent to the neighborhood filter of  $\infty$  in the one-point compactification of the  $\omega_1$ -sized discrete space, yet only the former point has an  $\omega_1$ -long escape sequence.

**Definition 2.6.** Given a preorder  $P$ ,

- $P$  is  $\kappa$ -directed if every  $A \in [P]^{<\kappa}$  has an upper bound in  $P$ ;
- $P$  is directed if it is  $\omega$ -directed;
- the *additivity*  $\text{Add}(P)$  of  $P$  is the least cardinal  $\lambda$  for which  $P$  is not  $\lambda$ -directed, if it exists;
- if  $P$  has a maximum, then  $\text{Add}(P) = \infty$ .

*Remark 2.7.*

- $\text{Add}(P)$  is always 1, 2,  $\infty$ , or a regular limit ordinal.
- $P$  is cofinally equivalent to a regular limit ordinal  $\mu$  if and only if  $\text{Add}(P) = \text{cf}(P) = \mu$ .
- If  $P \equiv_{\text{cf}} Q$ , then  $\text{Add}(P) = \text{Add}(Q)$  and  $\text{cf}(P) = \text{cf}(Q)$ .
- If  $f: P \rightarrow \kappa$  and  $\kappa < \text{Add}(P)$ , then  $f$  is constant on a cofinal subset of  $P$ .

**Lemma 2.8.** *If  $\kappa$  is a regular limit ordinal,  $D$  and  $E$  are preorders,  $X$  is a space,  $p \in X$ ,  $\text{Nbhd}(p, X) \equiv_{\text{cf}} D \times E$ , and  $\text{cf}(D) < \kappa < \text{Add}(E)$ , then  $\kappa \notin \text{Escape}(p, X)$ .*

*Proof.* We may assume  $\text{Nbhd}(p, X)$  and  $D \times E$  are disjoint, so there is a preordering  $\preceq$  of  $\text{Nbhd}(p, X) \cup (D \times E)$  that makes  $\text{Nbhd}(p, X)$  and  $D \times E$  cofinal suborders. Seeking a contradiction, suppose that  $\vec{U}$  is a  $\kappa$ -long escape sequence at  $p$ . It suffices to show that  $\vec{U}$  is eventually constant. Since  $\kappa < \text{Add}(E)$ , there exists  $e \in E$  such that for all  $i < \kappa$  there exists  $d \in D$  such that  $U_i \preceq (d, e)$ . For every  $d \in D$ , there exists  $i < \kappa$  such that  $(d, e) \preceq U_i$ . Since  $\text{cf}(D) < \text{Add}(\kappa)$ , it follows that, for some

$j < \kappa$ , every  $d \in D$  satisfies  $(d, e) \trianglelefteq U_j$ . Therefore,  $U_i \trianglelefteq U_j$  for all  $i < \kappa$ , so  $\vec{U}$  is eventually constant.  $\square$

*Remark 2.9.* The above proof works if we replace  $\text{Nbhd}(p, X)$  with an arbitrary preorder. In particular, if  $\kappa$  is regular limit ordinal,  $D$  and  $E$  are nonempty preorders, and  $\text{cf}(D) < \kappa < \text{Add}(E)$ , then  $D \times \kappa \times E \not\equiv_{\text{cf}} D \times E$  because  $D \times \kappa \times E$  has an unbounded increasing sequence of length  $\kappa$ .

**Lemma 2.10.** *If  $X$  is  $T_3$ ,  $p \in X$ ,  $D$  and  $E$  are preorders,  $\text{Nbhd}(p, X) \equiv_{\text{cf}} D \times E$ , and  $\text{cf}(D) < \text{Add}(E) < \infty$ , then  $\text{Add}(E) \in \text{Escape}_{\text{RO}}(p, X)$ .*

*Proof.* We may assume  $\text{Nbhd}(p, X)$  and  $D \times E$  are disjoint, so there is a preordering  $\trianglelefteq$  of  $\text{Nbhd}(p, X) \cup (D \times E)$  that makes  $\text{Nbhd}(p, X)$  and  $D \times E$  cofinal suborders. Since  $X$  is  $T_3$ , the regular open neighborhoods of  $p$  also form a cofinal suborder—call it  $\text{RO}(p, X)$ . For each  $(d, e) \in D \times E$ , let  $U(d, e)$  be the smallest (*i.e.*,  $\trianglelefteq$ -greatest) regular open neighborhood of  $p$  that contains (*i.e.*, is  $\trianglelefteq$ -below) every  $V \in \text{RO}(p, X)$  satisfying  $(d, e) \trianglelefteq V$ . For every  $(d, e) \in D \times E$ , there exist  $V \in \text{RO}(p, X)$  and  $(d', e') \in D \times E$  such that  $(d, e) \trianglelefteq V \trianglelefteq (d', e')$ , which implies  $(d, e) \trianglelefteq U(d', e')$ . So, choose  $(f, g): D \times E \rightarrow D \times E$  such that  $(d, e) \trianglelefteq U((f, g)(d, e))$  for all  $(d, e) \in D \times E$ .

By replacing  $D$  with a cofinal subset if necessary, we may assume that  $|D| = \text{cf}(D)$ . Fix  $d \in D$ . Since  $|D| < \text{Add}(E)$ , there is a cofinal subset  $E_0$  of  $E$  such that  $f(d, e) = f(d, e')$  for all  $e, e' \in E_0$ ; let  $b = f(d, e)$  for any  $e \in E_0$ . For each  $e \in E$ , let  $G(e) = g(d, e')$  for some  $e' \in E_0$  where  $e \leq e'$ ; we then have  $(d, e) \trianglelefteq U(b, G(e))$  for all  $e \in E_0$ . Set  $\kappa = \text{Add}(E)$ ; choose  $A = \{a_i : i < \kappa\} \subseteq E$  such that  $A$  is unbounded in  $E$ . Choose  $(e_i : i < \kappa)$  such that  $\{a_i, e_j, G(e_j)\} \leq e_i \in E_0$  for all  $j < i < \kappa$ . By construction,  $\vec{e}$  is increasing, so  $\vec{V} = (U(b, e_i) : i < \kappa)$  is increasing in  $\text{RO}(p, X)$ . Also by construction,  $\vec{e}$  is unbounded in  $E$  and  $(d, e_i) \trianglelefteq U(b, G(e_i)) \trianglelefteq U(b, e_{i+1})$  for all  $i < \kappa$ , so  $\vec{V}$  is unbounded in  $\text{RO}(p, X)$ . Thus,  $\vec{V}$  is a regular open escape sequence at  $p$  with length  $\text{Add}(E)$ .  $\square$

*Remark 2.11.* The above proof works if we replace the regular open neighborhoods of  $p$  with an arbitrary complete lattice with its top removed.

**Theorem 2.12.** *If  $\kappa_0 < \dots < \kappa_n$  are regular limit ordinals,  $X$  is  $T_3$ ,  $p \in X$ , and  $\text{Nbhd}(p, X) \equiv_{\text{cf}} \prod_{i < n} \kappa_i$ , then*

$$\text{Escape}(p, X) \cap \text{Reg} = \text{Escape}_{\text{RO}}(p, X) \cap \text{Reg} = \{\kappa_0, \dots, \kappa_n\}.$$

*Proof.* For each  $i \leq n$ ,  $\kappa_i \in \text{Escape}_{\text{RO}}(p, X)$  by Lemma 2.10 with  $D = \prod_{j < i} \kappa_j$  and  $E = \prod_{i \leq j \leq n} \kappa_j$ . (Note that  $\prod \emptyset = \{\emptyset\} = 1$ .) If  $\lambda$  is a regular limit ordinal not equal to any  $\kappa_i$ , then  $\lambda \notin \text{Escape}(p, X)$  by Lemma 2.8 with  $D = \prod(\{\kappa_i : i < n\} \cap \lambda)$  and  $E = \prod(\{\kappa_i : i < n\} \setminus \lambda)$ .  $\square$

Theorem 2.2 immediately follows from Theorems 2.4 and 2.12.

### 3. SKINNY RECTANGLES

To show that  $\omega \times \omega_2$  cannot be cofinally equivalent to every neighborhood filter of a compactum, we use free sequences. Recall that a transfinite sequence  $\vec{p}$  in a space  $X$  is *free* if  $\overline{\{p_j : j < i\}}$  and  $\overline{\{p_j : j \geq i\}}$  are disjoint for all  $i$ . Also, recall that  $\hat{F}(X)$  is the least cardinal  $\kappa$  such that  $X$  has no free sequence of length  $\kappa$ .

**Lemma 3.1.** *If  $X$  is a compactum,  $\alpha$  is a limit ordinal, and  $X$  has a free sequence of length  $\alpha$ , then  $X$  has an escape sequence of length  $\alpha$  at some point.*

*Proof.* Let  $\vec{p}$  be a free sequence of length  $\alpha$  in  $X$ . Choose  $q \in \overline{\bigcap_{i < \alpha} \{p_j : j \geq i\}}$ . Let  $U_i = X \setminus \overline{\{p_j : j < i\}}$  for each  $i < \alpha$ , so that  $\vec{U}$  is an escape sequence at  $q$ .  $\square$

**Corollary 3.2.** *If  $X$  is a compactum, then  $\bigcup_{p \in X} \text{Escape}(p, X)$  includes every infinite cardinal less than  $\hat{F}(X)$ .*

**Lemma 3.3.** *If  $X$  is a compactum,  $p \in X$ , and  $\kappa \in \text{Escape}(p, X) \cap \text{Reg}$ , then  $X$  has a free sequence of length  $\kappa$ .*

*Proof.* Let  $(U_i : i < \kappa)$  be an escape sequence at  $p$ . If we replace each  $U_i$  with its interior, then  $\vec{U}$  remains an escape sequence at  $p$ , so we may assume that each  $U_i$  is open. For each  $i < \kappa$ , choose a closed  $V_i \in \text{Nbhd}(p, X)$  such that  $V_i \subseteq U_i$ . For each  $\sigma \in [\kappa]^{<\omega}$ , there exists  $i < \kappa$  such that  $U_i \not\supseteq \bigcap_{j \in \sigma} V_j$ . Since  $\kappa$  is regular, we may assume that we have thinned out  $\vec{U}$  such that  $U_i \not\supseteq \bigcap_{j \in \sigma} V_j$  for all  $i < \kappa$  and  $\sigma \in [i]^{<\omega}$ . (Hence,  $((V_i, U_i) : i < \kappa)$  is free sequence of regular pairs in the sense of [22].) By compactness, there exists  $(x_i : i < \kappa)$  such that  $x_i \in \bigcap_{j \leq i} V_j \setminus U_{i+1}$ . Moreover,  $\vec{x}$  is free because  $\{x_j : j < i\}$  is contained in  $X \setminus U_i$  and  $\{x_j : j \geq i\}$  is contained in  $V_i$ .  $\square$

**Theorem 3.4.** *If  $X$  is a compactum and  $\kappa$  is a regular limit ordinal, then  $X$  has a neighborhood filter that is not cofinally equivalent to any  $D \times E$  where  $\text{cf}(D) < \kappa < \text{Add}(E) < \infty$ .*

*Proof.* Suppose that  $p \in X$  and  $\text{Nbhd}(p, X)$  is cofinally equivalent to some  $D_p \times E_p$  where  $\text{cf}(D_p) < \kappa \leq \text{Add}(E_p) < \infty$ . By Lemma 2.10,  $\text{Add}(E_p) \in \text{Escape}_{\text{RO}}(p, X)$ ; by Lemma 3.3,  $X$  has a free sequence of length  $\text{Add}(E_p)$ , so  $X$  has a free sequence of length  $\kappa$ . By Lemma 3.1,  $X$  has an escape sequence of length  $\kappa$  at some point  $q$ . By Lemma 2.8,  $\text{Nbhd}(q, X)$  is not cofinally equivalent to any  $D_q \times E_q$  where  $\text{cf}(D_q) < \kappa < \text{Add}(E_q)$ .  $\square$

**Corollary 3.5.** *Every compactum has a neighborhood filter that is not cofinally equivalent to  $\omega \times \omega_2$ .*

The remainder of this section is devoted to proving a stronger version of Corollary 3.2 for homogeneous compacta in models of GCH.

**Definition 3.6.** Let the *strict tightness*  $\hat{t}(X)$  of a space  $X$  be the least cardinal  $\kappa$  such that for every  $A \subseteq X$  and  $p \in \overline{A}$ , we have  $p \in \overline{B}$  for some  $B \in [A]^{<\kappa}$ . The *tightness*  $t(X)$  of  $X$  defined the same way, except that we replace  $[A]^{<\kappa}$  with  $[A]^{\leq \kappa}$ .

Lemma 3.7 is due to Arhangel'skiĭ and Shapirovskiĭ (see [2] or [19, Thm. 4.20]) for the case where  $\kappa$  is a successor cardinal (because the result is stated in terms of tightness, not strict tightness). However, as noted after Theorem 4.20 in [19], the proof clearly works for arbitrary regular infinite  $\kappa$ .

**Lemma 3.7.** *If  $X$  is a compactum and  $\kappa$  is a regular limit ordinal, then  $\hat{t}(X) \leq \kappa$  if and only if  $X$  has no free sequence of length  $\kappa$ .*

**Definition 3.8.** Given a space  $X$ ,  $\pi\chi(X) = \sup_{p \in X} \pi\chi(p, X)$ .

**Lemma 3.9** (Shapirovskiĭ). *If  $X$  is a compactum and  $p \in X$ , then  $X$  has a free sequence of length  $\pi\chi(p, X)$ .*

*Proof.* This lemma is just a localized form of Shapirovskii's Theorem,  $\pi\chi(X) \leq t(X)$  [20]. See the proof of a boolean algebraic version of Shapirovskii's Theorem in [19, Thm. 11.8]; it uses so-called free sequences of clopen sets and shows that our lemma holds if  $X$  is zero-dimensional. To adapt that proof to the general case, simply replace free sequences of clopen sets with Todorčević's notion of free sequences of regular pairs [22].  $\square$

**Definition 3.10.** The *weight*  $w(X)$  of  $X$  is the minimum of the cardinalities of bases of  $X$ .

The next lemma is due to De La Vega [24, Thm. 3.2], except that we extend it to handle the case where  $\lambda$  is weakly inaccessible.

**Lemma 3.11.** *If  $X$  is a homogeneous compactum,  $\lambda$  is a regular limit ordinal, and  $X$  has no free sequences of length  $\lambda$ , then  $|X| \leq 2^{<\lambda}$ .*

*Proof.* First, the lemma is trivial when  $\lambda = \omega$ . Second, by Lemma 3.7,  $\lambda \geq \hat{t}(X)$ . Therefore, if  $\lambda$  is a successor cardinal, then Theorem 3.2 of [24], which says  $|X| \leq 2^{t(X)}$ , implies  $|X| \leq 2^{<\lambda}$ . Finally, we can modify the proof of Theorem 3.2 of [24] to show that  $|X| \leq 2^{<\lambda}$  without assuming  $\lambda$  is a successor cardinal. In [24], Theorem 3.2 is deduced from Theorem 3.1, which assumes  $t(X) \leq \kappa$  and deduces  $w(X) \leq 2^\kappa$ , where  $\kappa$  is an arbitrary infinite cardinal. Thanks to regularity of  $\lambda$ , we may safely respectively replace “ $\leq \kappa$ ” and “ $2^\kappa$ ” with “ $< \lambda$ ” and “ $2^{<\lambda}$ ” throughout the statement and proof of Theorem 3.1. We also may safely replace all sequences and sets of size  $\kappa$  with sequences and sets of size less than  $\lambda$ , and replace “ $\kappa$ -closed” with “ $(< \lambda)$ -closed.” These simple changes yield a proof of  $w(X) \leq 2^{<\lambda}$ . Therefore, it suffices to show that  $|X| \leq w(X)^{<\lambda}$ .

To deduce  $|X| \leq 2^{t(X)}$  from  $w(X) \leq 2^{t(X)}$ , De La Vega uses two inequalities,  $|X| \leq w(X)^{\pi\chi(X)}$  and  $\pi\chi(X) \leq t(X)$ . The first of these inequalities is due to Van Mill [14] and it applies to all power homogeneous compacta. (Using the same kinds of cosmetic changes as in the previous paragraph, it is easy to check that his proof generalizes to show that if  $\pi\chi(p, X) < \lambda$  for all  $p \in X$ , then  $|X| \leq w(X)^{<\lambda}$ . However, since our  $X$  is homogeneous, we do not need to make these changes.) The second inequality localizes to Lemma 3.9, which implies  $\pi\chi(p, X) < \lambda$  for all  $p \in X$ . Hence,  $|X| \leq w(X)^{<\lambda}$ .  $\square$

**Definition 3.12.** Given a space  $X$  and  $A \subseteq X$ ,

- The *character*  $\chi(A, X)$  of  $A$  is the cofinality of the set of neighborhoods of  $A$  (ordered by  $\supseteq$ );
- the *pseudocharacter*  $\psi(A, X)$  of  $A$  is the minimum of the cardinalities of families of neighborhoods of  $A$  that have intersection  $A$ ;
- we abbreviate  $\chi(\{p\}, X)$  by  $\chi(p, X)$  and  $\psi(\{p\}, X)$  by  $\psi(p, X)$ ;
- $\chi(X) = \sup_{p \in X} \chi(p, X)$ .

It is easily checked that  $\chi(A, X) = \psi(A, X)$  whenever  $A$  and  $X$  are nested compacta.

The following lemma is due to Arhangel'skiĭ [1]. In [1], it is stated in terms of tightness. We state it as below for increased precision at limit cardinals.

**Lemma 3.13.** *If  $\lambda$  is an infinite cardinal,  $X$  is a compactum, and  $X$  has no free sequence of length  $\lambda$ , then  $X$  has a subset  $A$  of size less than  $\lambda$  such that  $\bar{A}$  has a nonempty closed subset  $B$  such that  $\psi(B, X) < \lambda$ .*

**Definition 3.14.**

- Given a space  $X$ ,  $\min\chi(X) = \min_{p \in X} \chi(p, X)$ .
- Given a cardinal  $\lambda$ ,  $\log \lambda = \min\{\kappa : \lambda \leq 2^\kappa\}$ .

The next lemma is due to Juhász [10]. Again, we state it differently for increased precision at limit cardinals.

**Lemma 3.15.** *If  $X$  is a compactum,  $\lambda$  is an infinite cardinal, and  $X$  has no free sequence of length  $\lambda$ , then  $\log(\min\chi(X)) < \lambda$ .*

**Corollary 3.16.** *If  $X$  is a compactum,  $\lambda$  is an infinite cardinal, and  $\lambda$  is not in  $\bigcup_{p \in X} \text{Escape}(p, X)$ , then  $\log(\min\chi(X)) < \lambda$ .*

**Lemma 3.17.** *If  $X$  is a space,  $p \in X$ , and  $\mu \in \text{Escape}(p, X) \cap \text{Reg}$ , then  $\chi(p, X) \geq \mu$ .*

*Proof.* If  $P$  is a preorder and  $\mu$  is a regular cardinal greater than  $\text{cf}(P)$ , then every map from  $\mu$  to  $P$  is bounded on a cofinal subset of  $\mu$ , so there is no unbounded increasing  $\mu$ -sequence in  $P$ .  $\square$

**Theorem 3.18 (GCH).** *If  $X$  is a homogeneous compactum and  $p \in X$ , then  $\text{Escape}(p, X)$  is a closed initial segment of the infinite cardinals with maximum  $\chi(p, X)$ .*

*Proof.* We may assume that  $\text{Escape}(p, X)$  is nonempty. Let  $\kappa$  be the supremum of  $\text{Escape}(p, X)$ . Let  $\lambda$  be the least infinite cardinal not in  $\text{Escape}(p, X)$ . By Lemma 3.17,  $\chi(p, X)$  is an upper bound of  $\text{Escape}(p, X) \cap \text{Reg}$ . Actually,  $\chi(p, X)$  bounds  $\text{Escape}(p, X) \setminus \text{Reg}$  too: by Arhangel'skii's Theorem,  $|X| \leq 2^{\chi(X)}$ , which implies  $|X| \leq 2^{\chi(p, X)}$  by homogeneity; hence,  $|\text{Nbhd}(p, X)| \leq \chi(p, X)^{++}$  by GCH. Thus,  $\kappa \leq \chi(p, X)$ . Therefore, it suffices to show that  $\chi(p, X) < \lambda = \kappa^+$ . By Lemmas 3.1 and 3.11,  $|X| \leq 2^{<\nu}$  where  $\nu$  is the least regular ordinal  $\geq \lambda$ . By the Čech-Pospišil Theorem,  $|X| \geq 2^{\min\chi(X)}$ , which implies  $|X| \geq 2^{\chi(p, X)}$  by homogeneity. Therefore, by GCH,  $\chi(p, X) < \nu$ , so  $\chi(p, X) \leq \lambda$ . Since  $\kappa \leq \chi(p, X)$ , it follows that  $\lambda$  is the least infinite cardinal strictly above every  $\mu \in \text{Escape}(p, X)$ . All that remains is to show that supremum of  $\text{Escape}(p, X)$  is attained, for if it is, then  $\nu = \lambda = \kappa^+$ . Seeking a contradiction, suppose that the supremum of  $\text{Escape}(p, X)$  is not attained. We then have that  $\kappa$  is a limit cardinal and  $\kappa = \lambda$ ; by GCH,  $\log(\kappa) = \kappa$ . By Corollary 3.16,  $\log(\min\chi(X)) < \lambda$ ; by homogeneity,  $\log(\chi(p, X)) < \lambda$ . Therefore,  $\log(\kappa) \leq \log(\chi(p, X)) < \lambda = \kappa = \log(\kappa)$ , which is absurd.  $\square$

De La Vega proved that GCH implies  $t(X) = \chi(X)$  for all homogeneous compacta [24]. Letting  $F(X)$  denote the supremum of the cardinalities of free sequences in  $X$ , we have  $F(X) = t(X)$  for all compacta, by Lemma 3.7. Theorem 3.18 allows us to deduce that the supremum  $F(X)$  is attained if GCH holds and  $X$  is a homogeneous compactum.

**Corollary 3.19 (GCH).** *If  $X$  is a homogeneous compactum, then  $\hat{F}(X) = \chi(X)^+$ .*

*Proof.* Fix  $p \in X$ . By Lemma 3.1 and homogeneity,  $\hat{F}(X) \leq \sup(\text{Escape}(p, X))^+$ . By Theorem 3.18,  $\max(\text{Escape}(p, X)) = \chi(p, X)$ , so  $\hat{F}(X) \leq \chi(p, X)^+$ . Moreover,  $\chi(p, X) < \hat{F}(X)$  by Lemma 3.3.  $\square$



It is easy to find inhomogeneous compacta  $X$  where the supremum  $F(X)$  is not attained,  $F(X) < \chi(X)$ , or both. For example, if  $X$  is the one-point compactification of the topological sum  $\bigoplus_{i < \aleph_{\aleph_{\omega_1}}} Y_i$  where  $Y_i$  is the ordered space  $\aleph_j$  where  $\aleph_{\aleph_j} \leq i < \aleph_{\aleph_{j+1}}$ , then  $\hat{F}(X) = F(X) = \aleph_{\omega_1}$  and  $\chi(X) = \aleph_{\aleph_{\omega_1}}$ .

#### 4. RECTANGLES IN $\beta\omega \setminus \omega$

**Definition 4.1.**

- A preorder is *cofinally rectangular* if it is cofinally equivalent to a finite product of linear orders.
- A preorder is *cofinally scalene* if it is cofinally equivalent to some  $\prod S$  where  $S \subseteq \text{Reg}$ .

We now turn to the class of neighborhood filters in the Stone-Ćech remainder  $\beta\omega \setminus \omega$ , focusing the properties of being cofinally scalene, a natural weakening of cofinally rectangular. Since  $\beta\omega \setminus \omega$  is not homogeneous [8], we are leaving behind Van Douwen's Problem, our initial motivation. However,  $\beta\omega \setminus \omega$  has been extensively studied, so if we are to examine cofinally scalene neighborhood filters for their own sake, then  $\beta\omega \setminus \omega$  is a reasonable place to start. (We will show that  $\beta\omega$  has no cofinally scalene nonprincipal neighborhood filters.)

It is essentially shown in [9] that  $\text{Nbhd}(p, \beta\omega \setminus \omega) \equiv_{\text{cf}} ([\mathfrak{c}]^{<\omega}, \subseteq)$  for some  $p \in \beta\omega \setminus \omega$ . (Extend any independent family of size  $\mathfrak{c}$  to an ultrafilter avoiding pseudointersections of infinite subsets of the independent family.) The next (easy) theorem implies that  $[\mathfrak{c}]^{<\omega}$  is not cofinally scalene.

**Theorem 4.2.** *If  $\kappa$  and  $\lambda$  are infinite cardinals,  $\kappa < \lambda$ , and  $\kappa$  is regular, then  $([\lambda]^{<\kappa}, \subseteq)$  is not cofinally scalene.*

*Proof.* Seeking a contradiction, suppose  $S \subseteq \text{Reg}$  and  $[\lambda]^{<\kappa} \cup \prod S$  has a preordering  $\trianglelefteq$  that makes  $[\lambda]^{<\kappa}$  and  $\prod S$  cofinal suborders. For each  $i < \kappa^+$ , choose  $f(i) \in \prod S$  such that  $\{i\} \trianglelefteq f(i)$ . Since  $\text{Add}(\prod S) = \text{Add}([\lambda]^{<\kappa}) = \kappa$ ,  $\kappa = \min(S)$ . Hence, there exist  $\alpha < \kappa$  and a cofinal subset  $I$  of  $\kappa^+$  such that  $f(i)(\kappa) = \alpha$  for all  $i \in I$ . Since  $\text{Add}(\prod(S \setminus \{\kappa\})) > \kappa$ ,  $f[J]$  is bounded for some  $J \in [I]^\kappa$ . Hence,  $[J]^1$  is bounded in  $[\lambda]^{<\kappa}$ ; this is our desired contradiction.  $\square$

Isbell's Problem asks whether it is consistent with ZFC that all neighborhood filters of  $\beta\omega$  are cofinally equivalent to  $[\mathfrak{c}]^{<\omega}$  or 1. In [18], it was shown that this is equivalent to asking whether it is consistent with ZFC that all neighborhood filters of  $\beta\omega \setminus \omega$  are cofinally equivalent to  $[\mathfrak{c}]^{<\omega}$ . Our next theorem solves an easier version of Isbell's Problem: consistently, there are no cofinally scalene neighborhood filters in  $\beta\omega \setminus \omega$ .

**Definition 4.3.**

- A point  $p$  in a space  $X$  is a *P-point* if  $\text{Nbhd}(p, X)$  is  $\omega_1$ -directed.
- A filter  $\mathcal{F}$  on  $\omega$  is assumed to be ordered by  $\supseteq$ , but  $\mathcal{F}_*$  denotes  $\mathcal{F}$  ordered by eventual containment  $\supseteq^*$ .
- $\beta\omega$  is identified with the space of ultrafilters on  $\omega$ .

Note that if  $\mathcal{U} \in \beta\omega \setminus \omega$ , then  $\mathcal{U} \equiv_{\text{cf}} \text{Nbhd}(\mathcal{U}, \beta\omega)$  and  $\mathcal{U}_* \equiv_{\text{cf}} \text{Nbhd}(\mathcal{U}, \beta\omega \setminus \omega)$ .

**Theorem 4.4.** *It is consistent with ZFC that no neighborhood filter in  $\beta\omega \setminus \omega$  is cofinally scalene.*

*Proof.* First,  $\beta\omega \setminus \omega$  has no isolated points, so we cannot have  $\text{Nbhd}(p, \beta\omega \setminus \omega) \equiv_{\text{cf}} 1$ . Second, by [18, Thm. 3.13], we cannot have  $\text{Nbhd}(p, \beta\omega \setminus \omega) \equiv_{\text{cf}} \omega \times D$  where  $D$  is  $\omega_1$ -directed. Third, there is a model of ZFC without P-points in  $\beta\omega \setminus \omega$  [21].  $\square$

*Remark 4.5.* For an alternative proof of the above theorem, force with finite binary partial functions on  $\kappa$  where  $\omega_1 < \kappa = \kappa^\omega$ . This yields a model of ZFC with P-points  $\mathcal{V}$  in  $\beta\omega \setminus \omega$  because  $\mathfrak{d} = \mathfrak{c}$  (see [6, Thm. 9.25]), but they all satisfy  $\mathcal{V}_* \equiv_{\text{cf}} [\mathfrak{c}]^{<\omega_1}$  because the generic sequence of Cohen reals has length  $\mathfrak{c}$  and none of its uncountable subsequences has an infinite pseudointersection. In this model,  $\mathfrak{c} = \kappa$ , so  $[\mathfrak{c}]^{<\omega_1}$  is not cofinally scalene by Theorem 4.2.

On the other hand, it is well known that  $\text{MA}(\sigma\text{-centered})$  implies that  $\mathfrak{c}$  is regular and  $\text{Nbhd}(p, \beta\omega \setminus \omega) \equiv_{\text{cf}} \mathfrak{c}$  for some  $p \in \beta\omega \setminus \omega$ . (See [6, Thms. 7.12, 7.14].) In [18], it was shown that  $\text{MA}(\sigma\text{-centered})$  also implies that for every regular limit ordinal  $\kappa \leq \mathfrak{c}$ ,  $\text{Nbhd}(p, \beta\omega \setminus \omega) \equiv_{\text{cf}} [\mathfrak{c}]^{<\kappa}$  for some  $p \in \beta\omega \setminus \omega$ . Are these all the cofinal types of neighborhood filters in  $\beta\omega \setminus \omega$  implied to exist by  $\text{MA}(\sigma\text{-centered})$ ? Does  $\text{MA}(\sigma\text{-centered})$  imply that every cofinally scalene neighborhood filter in  $\beta\omega \setminus \omega$  is cofinally equivalent to  $\mathfrak{c}$ ? The rest of this section develops some partial answers to these questions.

**Definition 4.6.**

- A map between directed sets is *Tukey* if it sends unbounded sets to unbounded sets.
- A map between directed sets is *convergent* if it sends cofinal sets to cofinal sets.
- $D \leq_T E$  means there is a Tukey map  $f: D \rightarrow E$ .
- $E \geq_T D$  means there a convergent map  $g: E \rightarrow D$ .

It is easy to check that  $D \leq_T E$  if and only if  $E \geq_T D$ . Tukey introduced the relation  $\geq_T$  in [23] and there proved that  $D \geq_T E \geq_T D$  if and only if  $D \equiv_{\text{cf}} E$ . (Tukey originally, equivalently defined  $E \geq_T D$  to mean that there exist maps  $f: D \rightarrow E$  and  $g: E \rightarrow D$  such that  $e \geq f(d) \Rightarrow g(e) \geq d$ .) The following lemma is implicit in [23].

**Lemma 4.7.** *Given directed sets  $A$ ,  $B$ , and  $C$ , we have  $A \times B \leq_T C$  if and only if  $A \leq_T C$  and  $B \leq_T C$ . In particular, if  $A \leq_T B$ , then  $B \equiv_{\text{cf}} A \times B$ .*

*Proof.* First, if  $h: A \times B \rightarrow C$  is Tukey, then, for any fixed  $(a_0, b_0) \in A \times B$ , the maps  $f(\bullet, b_0): A \rightarrow C$  and  $g(a_0, \bullet): B \rightarrow C$  are also Tukey. Second, if  $p: A \rightarrow C$  and  $q: B \rightarrow C$  are Tukey, then any  $r: A \times B \rightarrow C$  satisfying  $p(a), q(b) \leq r(a, b)$  is also Tukey. Third,  $B \leq_T A \times B$  is witnessed by  $b \mapsto (a_0, b)$  for any fixed  $a_0 \in A$ . Hence, if  $A \leq_T B$ , then  $A \times B \leq_T B \leq_T A \times B$ , which implies  $B \equiv_{\text{cf}} A \times B$   $\square$

**Definition 4.8.**

- Fix a pairing function  $\langle \bullet, \bullet \rangle$  from  $\omega \times \omega$  to  $\omega$ .
- For all  $E \subseteq \omega$  and  $i \in \omega$ ,  $E@i = \{j : \langle i, j \rangle \in E\}$ .
- Given filters  $\mathcal{F}, \mathcal{G}$  on  $\omega$ , the *Fubini product*  $\mathcal{F} \otimes \mathcal{G}$  is

$$\{E \subseteq \omega : \{i : E@i \in \mathcal{G}\} \in \mathcal{F}\}.$$

**Lemma 4.9 (CH).** *Every nonprincipal neighborhood filter in  $\beta\omega$  is cofinally equivalent to a neighborhood filter in  $\beta\omega \setminus \omega$ .*

*Proof.* Let  $\mathcal{U} \in \beta\omega \setminus \omega$  and let  $\mathcal{V}$  be P-point in  $\beta\omega \setminus \omega$ . It suffices to show that  $\mathcal{U} \equiv_{\text{cf}} (\mathcal{U} \otimes \mathcal{V})_*$ . The map  $E \mapsto \{(i, j) : (i, j) \in E \times \omega\}$  is Tukey from  $\mathcal{U}$  to  $(\mathcal{U} \otimes \mathcal{V})_*$ , so it suffices to show that  $(\mathcal{U} \otimes \mathcal{V})_* \leq_T \mathcal{U}$ . Every nonprincipal ultrafilter on  $\omega$  has uncountable cofinality. Hence, we can use CH to get a nondecreasing bijection  $h$  from  $\omega_1$  to  $\mathcal{U}$ . The map  $h$  is necessarily Tukey, so  $\omega_1 \times \mathcal{U} \equiv_{\text{cf}} \mathcal{U}$  by Lemma 4.7. Therefore, it suffices to show that  $(\mathcal{U} \otimes \mathcal{V})_* \leq_T \mathcal{U} \times \omega_1$ .

Define  $\pi: (\mathcal{U} \otimes \mathcal{V})_* \rightarrow \mathcal{U}$  by  $\pi(E) = \{i : E @ i \in \mathcal{V}\}$ . Let  $\psi: (\mathcal{U} \otimes \mathcal{V})_* \rightarrow \omega_1$  be an arbitrary injection. It suffices to show that  $(\pi, \psi)$  is Tukey. Since  $\psi$  sends uncountable sets to unbounded sets, it suffices to show that  $\pi$  sends countable unbounded sets to unbounded sets. We will prove the contrapositive. Suppose that  $A_n \in \mathcal{U} \otimes \mathcal{V}$  for all  $n < \omega$  and  $\{\pi(A_n) : n < \omega\}$  is bounded in  $\mathcal{U}$ . We need to show that  $\mathcal{A} = \{A_n : n < \omega\}$  is bounded in  $(\mathcal{U} \otimes \mathcal{V})_*$ . Since  $\pi[\mathcal{A}]$  is bounded,  $B = \bigcap \pi[\mathcal{A}] \in \mathcal{U}$ . Choose  $C \subseteq \omega$  such that  $C @ i = \emptyset$  for all  $i \notin B$  and, for all  $j \in B$ ,  $C @ j \in \mathcal{V}$ ,  $C @ j \subseteq \bigcap_{n \leq j} A_n @ j$ , and  $C @ j \subseteq^* A_n @ j$  for all  $n < \omega$ . We can find such  $C$  because  $\mathcal{V}$  is a P-point and  $A_n @ j \in \mathcal{V}$  for all  $(n, j) \in \omega \times B$ . It follows that  $C \in \mathcal{U} \otimes \mathcal{V}$  and  $C \subseteq^* A_n$  for all  $n < \omega$ , so  $\mathcal{A}$  is bounded in  $(\mathcal{U} \otimes \mathcal{V})_*$  as desired.  $\square$

**Theorem 4.10** (CH). *If  $X$  is  $\beta\omega \setminus \omega$  or  $\beta\omega$ , then there are  $2^{\mathfrak{c}}$  cofinal types of neighborhood filters in  $X$ , but all the cofinally scalene neighborhood filters in  $X$  are cofinally equivalent to  $\mathfrak{c}$  if  $X = \beta\omega \setminus \omega$ ; 1 if  $X = \beta\omega$ .*

*Proof.* By Theorem 31 of [7], MA( $\sigma$ -centered) implies that there are  $2^{\mathfrak{c}}$ -many P-points in  $\beta\omega \setminus \omega$  that, when ordered by  $\supseteq$ , are pairwise  $\leq_T$ -incomparable. Therefore, by CH, there are  $2^{\mathfrak{c}}$  cofinal types of neighborhood filters in  $\beta\omega$ . By Lemma 4.9, these  $2^{\mathfrak{c}}$  cofinal types are also instantiated by neighborhood filters in  $\beta\omega \setminus \omega$ .

For the second half of the theorem, fix  $\mathcal{U} \in \beta\omega \setminus \omega$ . By Theorems 3.13 and 3.16 of [18],  $\mathcal{U}_* \not\equiv_{\text{cf}} \omega \times D$  and  $\mathcal{U} \not\equiv_{\text{cf}} \omega \times D$  if  $\omega < \text{Add}(D)$ . Therefore, if  $\mathcal{U}$  is not a P-point in  $\beta\omega \setminus \omega$ , then  $\mathcal{U}_*$  is not cofinally scalene. If  $\mathcal{U}$  is a P-point in  $\beta\omega \setminus \omega$ , then  $\mathcal{U}_* \equiv_{\text{cf}} \mathfrak{c}$  by CH. Finally,  $\mathcal{U}$  is not a P-point in  $\beta\omega$  because it is in the closure of  $\omega$ , so  $\mathcal{U}$  is not cofinally scalene, but every principal ultrafilter is.  $\square$

*Remark 4.11.* The proof that all cofinally scalene neighborhood filters in  $\beta\omega$  are principal did not use CH.

For cofinally scalene neighborhood filters in  $\beta\omega \setminus \omega$ , CH is a relatively uninteresting context because the only possible cofinality is  $\omega_1$ , so the only possible cofinal types are  $\omega_1$  and  $\omega \times \omega_1$ . Our next theorem's hypotheses allow  $\mathfrak{c}$  to be arbitrarily large.

**Definition 4.12.** Given a class  $\Gamma$  of forcings, let  $\mathfrak{m}_\Gamma$  denote the least cardinal  $\kappa$  such that there is a forcing  $\mathbb{P} \in \Gamma$  and a family  $\mathcal{D}$  of  $\kappa$ -many dense subsets of  $\mathbb{P}$  such that no filter of  $\mathbb{P}$  meets every  $D \in \mathcal{D}$ .

**Lemma 4.13.** *If  $1 \leq n < \omega$ ,  $\kappa$  is a regular limit ordinal,  $\kappa < \mathfrak{m}_{\sigma-n\text{-linked}}$ , and  $\kappa < \text{Add}(D)$ , then no neighborhood filter in  $\beta\omega \setminus \omega$  is cofinally equivalent to  $\kappa \times D$ .*

*Proof.* The proof goes like the author's proof of Theorem 3.13 of [18], which handled the case where  $n = 1$  and  $\kappa = \omega$ . (Note that  $\omega_1 = \mathfrak{m}_{\sigma-1\text{-linked}}$  because all forcings are 1-linked.) Fix  $\mathcal{U} \in \beta\omega \setminus \omega$ . Seeking a contradiction, suppose that  $\sqsubseteq$  is a preordering of  $\mathcal{U} \cup (\kappa \times D)$  that makes  $\mathcal{U}_*$  and  $\kappa \times D$  cofinal suborders. We then have  $\text{Add}(\mathcal{U}_*) = \kappa$ . Therefore, we can find  $(F(i, a) : i < \kappa)$  such that  $(i, a) \sqsubseteq F(i, a) \in \mathcal{U}$  and  $F(j, a) \supseteq^* F(i, a)$  for all  $j < i < \kappa$ . Next, for each  $d \in D \setminus \{a\}$ , choose

$(F(i, d) : i < \kappa)$  such that  $(i, d) \leq F(i, d) \in \mathcal{U}$  and  $F(j, a) \supseteq^* F(i, d)$  for all  $j \leq i < \kappa$ .

Inductively construct  $g: \kappa \rightarrow \kappa$  as follows. Given  $i < \kappa$  and  $g \upharpoonright i$ , if  $d \in D$ , then  $\{F(g(j), d) : j < i\} \leq (l, b)$  for some  $l < \kappa$  and  $b \in D$ . Since  $\kappa < \text{Add}(D)$ , there exist  $l < \kappa$  and a cofinal subset  $C_i$  of  $D$  such that for all  $c \in C_i$  there exists  $b \in D$  such that  $\{F(g(j), c) : j < i\} \leq (l, b)$ . Choose  $g(i) \geq l$  such that  $g(j) < g(i)$  for all  $j < i$ . This completes the construction of  $g$ . Observe that  $g$  is strictly increasing and therefore has range cofinal in  $\kappa$ .

Set  $m = n + 1$ . For each  $j < m$ , let  $I_j$  be the ideal generated by  $\{F(g(2mi + 2j), a) \setminus F(g(2mi + 2j + 2), a) : i < \kappa\}$ . Observe that  $X \cap Y$  is finite for all  $X \in I_s$  and  $Y \in I_t$  where  $s < t < m$ . Since  $\kappa < \mathfrak{m}_{\sigma-n\text{-linked}}$ , Corollary 21 of [4] implies that there exist  $A_0, \dots, A_{m-1} \subseteq \omega$  such that  $\bigcap_{j < m} A_j = \emptyset$  and, for all  $j < m$  and  $X \in I_j$ ,  $X \subseteq^* A_j$ . Choose  $j < m$  such that  $A_j \notin \mathcal{U}$ ; choose  $\alpha < \kappa$  and  $d \in D$  such that  $\omega \setminus A_j \leq (\alpha, d)$ . Choose  $i < \kappa$  such that  $\alpha \leq g(2mi + 2j)$ ; choose  $c \in C_{2mi+2j+1}$  such that  $d \leq c$ . Since  $\omega \setminus A_j \leq (g(2mi + 2j), c)$ , we have  $\omega \setminus A_j \supseteq^* F(g(2mi + 2j), c)$ . Since  $F(g(2mi + 2j), a) \supseteq^* F(g(2mi + 2j), c)$ , we have  $F(g(2mi + 2j), a) \setminus A_j \supseteq^* F(g(2mi + 2j), c)$ . Hence,  $F(g(2mi + 2j + 2), a) \supseteq^* F(g(2mi + 2j), c)$ , which implies  $(2mi + 2j + 2, a) \leq F(g(2mi + 2j), c)$ . By our choice of  $c$ ,  $F(g(2mi + 2j), c) \leq (2mi + 2j + 1, b)$  for some  $b \in D$ . Therefore,  $(2mi + 2j + 2, a) \leq (2mi + 2j + 1, b)$ . We now have our desired contradiction:  $(2mi + 2j + 2, a) \leq (2mi + 2j + 1, b)$  but  $2mi + 2j + 2 \not\leq 2mi + 2j + 1$ .  $\square$

*Remark 4.14.* By Theorem 24 of [4] and the fact that  $\mathfrak{m}_{\sigma\text{-centered}}$  is regular (see [6, Thms. 7.12, 7.14]), it is consistent with ZFC that

$$\mathfrak{m}_{\sigma-n\text{-linked}} < \sup_{1 \leq k < \omega} \mathfrak{m}_{\sigma-k\text{-linked}} < \mathfrak{m}_{\sigma\text{-centered}}$$

for all  $n \in [1, \omega)$ . Also, the proof of Theorem 5.10 of [17] shows that, given any pair  $(\kappa, \lambda)$  of uncountable regular cardinals satisfying  $\lambda = \lambda^{<\kappa}$ , some  $\mathbb{P}$  satisfying Knaster's condition forces  $\mathfrak{m}_{\text{countable}} = \mathfrak{m}_{\sigma\text{-centered}} = \kappa$ ,  $\mathfrak{c} = \lambda$ , and the existence of a neighborhood filter of  $\beta\omega \setminus \omega$  that is cofinally equivalent to  $\kappa \times \lambda$ .

**Theorem 4.15.** *If  $\text{MA}(\sigma\text{-}n\text{-linked})$  holds for some  $n \in [1, \omega)$ , then every cofinally scalene neighborhood filter in  $\beta\omega \setminus \omega$  is cofinally equivalent to  $\mathfrak{c}$ .*

*Proof.* Fix  $\mathcal{U} \in \beta\omega \setminus \omega$ . It is well known that  $\text{MA}(\sigma\text{-}n\text{-linked})$  implies  $\text{MA}(\text{countable})$  implies  $\text{cf}(\mathcal{U}_*) = \mathfrak{c}$  (see [6, Thms. 7.13, 5.19, 9.7]), so if  $\mathcal{U}_*$  is cofinally scalene, then  $\mathcal{U}_* \equiv_{\text{cf}} \prod S$  where  $\emptyset \neq S \subseteq \text{Reg} \cap [\omega, \mathfrak{c}]$ . By Lemma 4.13,  $\min(S) \geq \mathfrak{m}_{\sigma-n\text{-linked}}$ , so  $S = \{\mathfrak{c}\}$ .  $\square$

*Remark 4.16.*  $\text{MA}(\sigma\text{-}1\text{-linked})$  is equivalent to CH.

*Question 4.17.* Can the hypothesis of the above theorem can be weakened to  $\text{MA}(\sigma\text{-centered})$ ?  $\text{MA}(\text{countable})$ ?

## 5. PRODUCTS AND FUBINI PRODUCTS

The proof of Lemma 4.9 used Fubini products to partially answer questions about the frequency of cofinally scalene posets of the form  $\mathcal{U}_*$  where  $\mathcal{U} \in \beta\omega \setminus \omega$ : CH implies that there are  $2^{\mathfrak{c}}$  cofinal types of the form  $\mathcal{U}_* / \equiv_{\text{cf}}$ , but only one is cofinally scalene. Working in the opposite direction, we now use product orders to answer a question about the cofinal types of Fubini products. Given a filter  $\mathcal{F}$  on  $\omega$ , adopt the notation

$\mathcal{F}^{\otimes 1} = \mathcal{F}$  and  $\mathcal{F}^{\otimes n+1} = \mathcal{F} \otimes \mathcal{F}^{\otimes n}$ , noting that the Fubini product is associative modulo the order isomorphisms induced by  $\langle\langle i, j \rangle, k \rangle \leftrightarrow \langle i, \langle j, k \rangle \rangle$ . Question 39 of [7] asks whether there is an ultrafilter  $\mathcal{U}$  on  $\omega$  such that  $\mathcal{U} <_T \mathcal{U}^{\otimes 2} <_T \mathcal{U}^{\otimes 3} <_T [\mathfrak{c}]^{<\omega}$ . Our answer is a strong “no.”

**Lemma 5.1.** *If  $\mathcal{F}$  and  $\mathcal{G}$  are nonprincipal filters on  $\omega$ , then  $\mathcal{F} \otimes \mathcal{G} \equiv_{\text{cf}} \mathcal{F} \times \mathcal{G}^\omega$ .*

*Proof.* By Lemma 4.7, it suffices to show that (1)  $\mathcal{F} \leq_T \mathcal{F} \otimes \mathcal{G}$ , (2)  $\mathcal{G}^\omega \leq_T \mathcal{F} \otimes \mathcal{G}$ , and (3)  $\mathcal{F} \otimes \mathcal{G} \leq_T \mathcal{F} \times \mathcal{G}^\omega$ . First, the map  $E \mapsto \{\langle i, j \rangle : (i, j) \in E \times \omega\}$  is Tukey from  $\mathcal{F}$  to  $\mathcal{F} \otimes \mathcal{G}$ . Second, let us construct a convergent map  $\Phi: \mathcal{F} \otimes \mathcal{G} \rightarrow \mathcal{G}^\omega$ . Given  $A \in \mathcal{F} \otimes \mathcal{G}$  and  $i < \omega$ , set  $\pi(A) = \{j : A @ j \in \mathcal{G}\}$  and  $\Phi(A)(i) = A @ (\min(\pi(A) \setminus i))$ . Suppose that  $\mathcal{C} \subseteq \mathcal{F} \otimes \mathcal{G}$  is cofinal and  $\xi \in \mathcal{G}^\omega$ . To prove that  $\Phi$  is convergent, it suffices to show that  $\Phi(A) \geq \xi$  for some  $A \in \mathcal{C}$ . Set  $\zeta(i) = \bigcap_{j \leq i} \xi(j)$  for all  $i < \omega$ . Choose  $A \in \mathcal{C}$  such that  $A \subseteq \bigcup_{i < \omega} (\{i\} \times \zeta(i))$ . Then, for all  $i < \omega$ ,

$$\Phi(A)(i) \subseteq \zeta(\min(\pi(A) \setminus i)) \subseteq \zeta(i) \subseteq \xi(i),$$

so  $\Phi(A) \geq \xi$  as desired.

Finally, we construct a convergent map  $\Psi: \mathcal{F} \times \mathcal{G}^\omega \rightarrow \mathcal{F} \otimes \mathcal{G}$ . Given  $(A, \xi) \in \mathcal{F} \times \mathcal{G}^\omega$ , set  $\Psi(A, \xi) = \bigcup_{i \in A} (\{i\} \times \xi(i))$ . The map  $\Psi$  is surjective and order preserving, so it is convergent.  $\square$

**Theorem 5.2.** *For all nonprincipal filters  $\mathcal{F}, \mathcal{G}$  on  $\omega$ ,  $\mathcal{F} \otimes \mathcal{G} \equiv_{\text{cf}} \mathcal{F} \otimes \mathcal{G}^{\otimes 2}$ . In particular,  $\mathcal{F}^{\otimes 2} \equiv_{\text{cf}} \mathcal{F}^{\otimes 3}$ .*

*Proof.* Use Lemma 5.1 three times.

$$\begin{aligned} \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{G}) &\equiv_{\text{cf}} \mathcal{F} \times (\mathcal{G} \otimes \mathcal{G})^\omega \\ &\equiv_{\text{cf}} \mathcal{F} \times (\mathcal{G} \times \mathcal{G}^\omega)^\omega \\ &\equiv_{\text{cf}} \mathcal{F} \times \mathcal{G}^\omega \\ &\equiv_{\text{cf}} \mathcal{F} \otimes \mathcal{G}. \end{aligned}$$

$\square$

As another application of Lemma 5.1, we show that the Fubini product is commutative modulo cofinal equivalence among the nonprincipal P-filters on  $\omega$  (i.e., nonprincipal filters  $\mathcal{F}$  on  $\omega$  such that if  $\vec{A} \in \mathcal{F}^\omega$ , then, for some  $B \in \mathcal{F}$ ,  $B \subseteq^* A_i$  for all  $i < \omega$ ).

**Theorem 5.3.** *If  $\mathcal{F}$  and  $\mathcal{G}$  are nonprincipal P-filters on  $\omega$ , then  $\mathcal{F} \otimes \mathcal{G} \equiv_{\text{cf}} \mathcal{G} \otimes \mathcal{F}$ .*

*Proof.* By Lemma 5.1, it suffices to show that  $\mathcal{F}^\omega \equiv_{\text{cf}} \mathcal{F} \times \omega^\omega$ . By Lemma 4.7, it suffices to show that (1)  $\mathcal{F} \leq_T \mathcal{F}^\omega$ , (2)  $\omega^\omega \leq_T \mathcal{F}^\omega$ , and (3)  $\mathcal{F}^\omega \leq_T \mathcal{F} \times \omega^\omega$ . First, the diagonal map from  $\mathcal{F}$  to  $\mathcal{F}^\omega$  is Tukey. Second,  $(n_i : i < \omega) \mapsto (\omega \setminus n_i : i < \omega)$  is a Tukey map from  $\omega^\omega$  to  $\mathcal{F}^\omega$ . Finally, following the proof of Theorem 33 in [7], map each  $\vec{A} \in \mathcal{F}^\omega$  to some  $(B, h) \in \mathcal{F} \times \omega^\omega$  such that  $B \setminus h(i) \subseteq A_i$  for all  $i < \omega$ . Again, the map is Tukey.  $\square$

All the results of this section naturally generalize to the  $\kappa$ -complete uniform filters on an arbitrary infinite regular  $\kappa$ .

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