Applications of ω_1-approximation systems

David Milovich

Boise Extravaganza in Set Theory

March 28, 2008

Notation. H_κ is the set of sets that are hereditarily smaller than κ.

Convention. A set M is a submodel if $\langle M, \in \rangle$ an elementary substructure of $\langle H_\theta, \in \rangle$ for a sufficiently large regular cardinal θ.

- Suppose we need to construct an object O of size $\aleph_1$ one countable piece at a time.

- At stage α, we may also need the partially completed object $O_{<\alpha}$ to satisfy certain closure properties.

- A elementary chain of countable submodels $\langle M_\alpha \rangle_{\alpha<\omega_1}$ is the natural tool to use.

- An arbitrarily chosen $\langle M_\alpha \rangle_{\alpha<\omega_1}$ might not work. To ensure $O_{<\alpha} \in M_\alpha$, we may need to construct $\langle O_\alpha \rangle_{\alpha<\omega_1}$ and $\langle M_\alpha \rangle_{\alpha<\omega_1}$ simultaneously: $M_0, O_0, M_1, O_1, M_2, O_2, \ldots$

- What if we need to construct an object O of size $\aleph_2$ one countable piece at a time?

- An elementary chain of $\aleph_1$-sized submodels $\langle M_\alpha \rangle_{\alpha<\omega_2}$ won't work because in general $|M_{\alpha+1} \setminus M_\alpha| = \aleph_1$.

- Okay, so we'll build $M_{\alpha+1}$ as the union of an elemenatry chain $\langle N_{\alpha,\beta} \rangle_{\beta<\omega_1}$ of countable submodels. With each $N_{\alpha,\beta}$ we'll build another countable piece $O_{\alpha,\beta}$ of O.

- Possible pitfall #1: we may assume $O_{<\alpha}, M_\alpha, O_{\alpha,<\beta} \in N_{\alpha,\beta}$ and $O_{\alpha,<\beta} \subseteq N_{\alpha,\beta}$ but not $O_{<\alpha} \subseteq N_{\alpha,\beta}$ or $M_\alpha \subseteq N_{\alpha,\beta}$.

- Possible pitfall #2: $O_{<\alpha}$ and $O_{\alpha,<\beta}$ might interact in a nasty way.

- If we can get around the pitfalls, we've got something useful.

- Okay, that covers $\aleph_2$, but why stop there? It's easy to similarly handle $\aleph_3$ and $\aleph_4$, and not terribly hard to see how to handle $\aleph_\omega$ and $\aleph_{\omega+1}$. We just need to formalize how to handle an arbitary cardinal.

Definition. A sequence of countable submodels $\langle M_\alpha \rangle_{\alpha<\eta}$ is an ω_1-approximation sequence if $\langle M_\beta \rangle_{\beta<\alpha} \in M_\alpha$ for all $\alpha \leq \eta$.

Theorem. There is a $\emptyset$-definable map Ψ with domain equal to the class of ω_1-approximation sequences, such that every $\Psi(\langle M_\alpha \rangle_{\alpha<\eta})$ is of the form $\langle \Sigma_\alpha \rangle_{\alpha\leq\eta}$ where, for all $\alpha \leq \eta$:

- Σ_α is a finite set of submodels.
- $\bigcup \Sigma_\alpha = \bigcup_{\beta<\alpha} M_\beta$.
- if $\alpha < \eta$, then $N \in M_\alpha$ for all $N \in \Sigma_\alpha$.

We call $\langle \Sigma_\alpha \rangle_{\alpha\leq\eta}$ an ω_1-approximation system.

About the proof

- Let Ω denote the class of finite, nonempty ordinal sequences $\langle \gamma_i \rangle_{i<n}$ for which $|\gamma_0| > |\gamma_1| > \cdots > |\gamma_{n-2}| \geq \aleph_1 > |\gamma_{n-1}|$ if $n \geq 2$ and $\aleph_1 > |\gamma_{n-1}|$ if $n = 1$.

- Let $\sqsubseteq$ denote the lexicographic ordering of Ω. Since $\sqsubseteq$ is a well-ordering, there is a unique isomorphism Υ from the ordinals to $\langle \Omega, \sqsubseteq \rangle$.

- Suppose $\Upsilon(\alpha) = \langle \gamma_i \rangle_{i<n}$. For each $k < n-1$, set $N_{\alpha,k} = \bigcup\{M_\beta : \langle \gamma_0, \ldots, \gamma_{k-1}, 0 \rangle \sqsubseteq \Upsilon(\beta) \sqsubset \langle \gamma_0, \ldots, \gamma_k, 0 \rangle\}$. Set $N_{\alpha,n-1} = \bigcup\{M_\beta : \langle \gamma_0, \ldots, \gamma_{n-2}, 0 \rangle \sqsubseteq \Upsilon(\beta) \sqsubset \langle \gamma_0, \ldots, \gamma_{n-1} \rangle\}$. Set $\Sigma_\alpha = \{N_{\alpha,j} : j < n\} \setminus \{\emptyset\}$. This works.

Applications

- **Theorem** (Jackson and Mauldin, 2002). There is a Steinhaus set, that is, a subset S of $\mathbb{R}^2$ that meets every isometric copy of $\mathbb{Z}^2$ at exactly one point.

- Jackson and Mauldin didn't exactly use ω_1-approximation systems. They more directly used the tree T formed by the closure of Ω with respect to initial segments. They built a sequence of substructures of $\mathbb{R}^2$ indexed by a subtree of T.

- **Definition.** A poset P is is κ^{op}*-like* if no element of P has κ-many elements above it.

- **Definition.** A base of a boolean algebra B is a subset A of B such that for all $b \in B$ we have $b = \bigvee_{i<n} a_i$ for some $a_0, \ldots, a_{n-1} \in A$.

- **Definition.** A boolean algebra is free if it isomorphic to the algebra of clopen sets $\mathrm{Clop}(2^\lambda)$ for some cardinal λ.

- **Definition.** Let the reaping number $\mathfrak{r}(B)$ of a boolean algebra B be the least κ such that some ultrafilter U of B has a refinement of size κ.

Lemma. Free boolean algebras reflect cones. More precisely, if A is a free boolean algebra, M is a submodel, and $a \in A \in M$ (but maybe $a \notin M$), then $\{b \in A \cap M : a < b\}$ has a minimum element.

Theorem. Every subset of a free boolean algebra contains a dense ω^{op}-like subset.

Proof. Use the lemma, elementary chains, and induction on the size of the boolean algebra.

Theorem. If B is a subalgebra of a free boolean algebra and $\mathfrak{r}(B) = |B|$, then every base of B contains an ω^{op}-like base.

Proof. Use the lemma and ω_1-approximation systems.

Topological consequences

Definition. A base $\mathcal{A}$ of a space X is a family of open sets such that for every open U and every $p \in U$ there exists $V \in \mathcal{A}$ such that $p \in V \subseteq U$.

Definition. A local base $\mathcal{A}$ at a point p in a space of X is a family of open sets containing p such that for every open U containing p there exists $V \in \mathcal{A}$ such that $V \subseteq U$.

Definition. A π-base $\mathcal{A}$ of a space X is a family of nonempty open sets such that for every nonempty open U there exists $V \in \mathcal{A}$ such that $V \subseteq U$.

Definition. A dyadic compactum is a Hausdorff space that is a continuous image of 2^λ for some λ. (Example: compact groups are dyadic.)

Convention. Families of open sets are ordered by $\subseteq$.

Theorem. Let X be a dyadic compactum. Then every π-base of X contains an ω^{op}-like π-base and every local base in X contains an ω^{op}-like local base.

Theorem. If X is also homogeneous, then every base of X contains an ω^{op}-like base.

λ-approximation systems

- Suppose $\lambda = \mathrm{cf}(\lambda) > \omega$. Replace $\aleph_1$ with λ in the definition of Ω. Replace "M_α countable" with "$|M_\alpha| \subseteq M_\alpha \cap \lambda \in \lambda$". Then you've got λ-approximation systems.

Theorem. Let C be a coproduct of boolean algebras each no bigger than κ. Every subset of C contains a κ^{op}-like dense subset.
Proof. Use elementary chains.

Theorem. If B is a subalgebra of C and $\mathfrak{r}(B) = |B|$, then every base of B contains a κ^{op}-like base.
Proof. Use κ^+-approximation systems and ω_1-approximation systems.

Topological consequences

- Let X be a Hausdorff space and a continuous image of a product of compact Hausdorff spaces each with weight at most κ.

- Every π-base of X contains an κ^{op}-like π-base and every local base in X contains an κ^{op}-like local base.

- If X is also homogeneous, then every base of X contains a κ^{op}-like base.

Questions

- If B is a boolean algebra and B reflects cones, then is B a subalgebra of a free boolean algebra?

- The answer is yes if $|B| \leq \aleph_1$. I haven't figured out how to avoid Pitfall #2.

- Is there a topological space X with a κ^{op}-like base $\mathcal{A}$ and a base $\mathcal{B}$ that does not contain a κ^{op}-like base? (This can't happen for π-bases or for local bases at a common point.)

- The answer is no if we have GCH and X compact, Hausdorff, and homogeneous.

References

S. Jackson and R. D. Mauldin, *On a lattice problem of H. Steinhaus*, J. Amer. Math. Soc. **15** (2002), no. 4, 817–856.