A team-game characterization of Cohen forcing

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Co-absolutes of powers of 2

Topological Problem. Characterize the spaces co-absolute with powers of 2.

- The regular open algebra $\mathrm{RO}(X)$ of a space is the Boolean algebra consisting of of regular open subsets of X ordered by inclusion.
- This Boolean algebra is complete: a union of regular opens has a smallest regular open superset.
- The Stone dual $\mathrm{Ult}(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ is the space of ultrafilters of $\mathbb{B}$ with basic clopen sets of the form $\{U \in \mathbb{B} \mid a \in U\}$.
- The absolute of a space X is the Stone dual $\mathrm{Ult}(\mathrm{RO}(X))$ of its regular open algebra.
- Spaces X and Y are co-absolute if their absolutes are homeomorphic.

Forcings equivalent to Cohen forcing

Forcing Problem. Characterize the posets forcing equivalent with Cohen forcings.

- Give a poset $\mathbb{P}$ the lower topology with basic open sets $\{q \mid q \leq p\}$.
- The Boolean completion $\overline{\mathbb{P}}$ of $\mathbb{P}$ is its regular open algebra $\mathrm{RO}(\mathbb{P})$. Each $p \in \mathbb{P}$ is identified with the smallest regular open superset of $\{q \mid q \leq p\}$.
- Posets $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent if every $\mathbb{P}$-generic forcing extension is a $\mathbb{Q}$-generic forcing extension and conversely.
- Posets $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent iff they are co-complete, that is, their Boolean completions are Boolean isomorphic.
- A Cohen forcing $\mathrm{Fn}(\kappa, 2)$ consists of the of finite binary partial functions on some infinite cardinal κ ordered by containment.

Cohen algebras

Algebra Problem. Characterize the Boolean algebras that are Cohen.

- The completion $\overline{\mathbb{B}}$ of a Boolean algebra $\mathbb{B}$ is the regular open algebra $\mathrm{RO}(\mathbb{B} \setminus \{0\})$ of the poset $\mathbb{B} \setminus \{0\}$.
- Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ are co-complete or forcing equivalent if their completions are Boolean isomorphic.
- A Boolean algebra is free if it is Boolean isomorphic to the clopen algebra of some power of 2.
- A Boolean algebra is free iff it is the finite joins of a dense subset order isomorphic to some $\mathrm{Fn}(\kappa, 2)$
- A Boolean algebra is Cohen if it is co-complete with some infinite free Boolean algebra.

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Team games

Suppose G is a game where I and II take turns playing sequences of fixed length τ.

$$\begin{array}{llllll} \mathrm{I} & p_0^0, \ldots, p_{\tau-1}^0 & & p_0^1, \ldots, p_{\tau-1}^1 & & \cdots \\ \mathrm{II} & & q_0^0, \ldots, q_{\tau-1}^0 & & q_0^1, \ldots, q_{\tau-1}^1 & \cdots \end{array}$$

- Call I a team of size τ.
- Call $(p_i^0, p_i^1, p_i^2, \ldots)$ the plays of player I_i.
- Call a strategy σ for team I uncoordinated if each p_i^n played according to σ depends only on q_i^m for $m < n$.
- In other words, each player I_i following an uncoordinated strategy for I ignores I_j and II_j for $j \neq i$.
- Analogously define uncoordinated strategies for II.

Example: the Banach-Mazur game for teams

Let $X = \prod_{i<\tau} X_i$ be a nonempty topological product space, $A \subset X$, and $1 \leq \tau < \omega$.

- For each $i < \tau$, I_i and II_i play open subsets of X_i

I	U_i^0		U_i^1		$\cdots$
II		V_i^0		V_i^1	$\cdots$

  such that $U_i^0 \supset V_i^0 \supset U_i^1 \supset V_i^1 \supset \cdots$.
- II wins iff $\bigcap_{n<\omega} \prod_{i<\tau} V_i^n \subset A$.
- II has a winning strategy iff A is comeager.
- II has an uncoordinated winning strategy iff A contains a product of τ comeager sets.
- If $\tau \geq 2$ and $A = \{x \in \mathbb{R}^\tau \mid \sum_{i<\tau} x_i \notin \mathbb{Q}\}$, then II has a winning strategy iff they coordinate.

The open-open game for teams

Given a topological space X and finite team size τ:

- In round $n < \omega$, team I plays open sets $(U_i^n)_{i<\tau}$.
- Then team II plays open sets $(V_i^n)_{i<\tau}$ such that $\bigcap_i V_i^n \subset \bigcap_i U_i^n$.
- II must also satisfy $\bigcap_i U_i^n \neq \varnothing \Rightarrow \bigcap_i V_i^n \neq \varnothing$.
- I wins iff $\bigcup_n \bigcap_i V_i^n$ is dense.

This a team version of the Daniel-Kunen-Zhou open-open game.

Game Problem. Characterize the spaces X for which team I has an uncoordinated winning strategies for the open-open game for all finite team sizes.

To define the team open-open game for a poset $\mathbb{P}$, restrict I and II to play elements of $\mathbb{P}$ and identify these with elements of $\mathrm{RO}(\mathbb{P})$.

The regular subalgebra game for teams

Given a Boolean algebra $\mathbb{B}$ and finite team size τ:

- Teams I and II play elements of $\mathbb{B}$ for ω rounds.

I	$(x_i^0)_{i<\tau}$		$(x_i^1)_{i<\tau}$		$\cdots$
II		$(y_i^0)_{i<\tau}$		$(y_i^1)_{i<\tau}$	$\cdots$

- Let S be the set of all plays $\bigcup_{i<\tau}\bigcup_{n<\omega}\{x_i^n, y_i^n\}$.
- Let $\langle S\rangle$ be the Boolean subalgebra generated by S.
- II wins iff $\langle S\rangle$ is a regular subalgebra of $\mathbb{B}$.
  - This is defined on the next slide.

This a team version of a game of Balcar, Jech, and Zapletal.

- To make this a poset game, play elements of a poset $\mathbb{P}$ and say that II wins iff $\langle S\rangle$ is a regular subalgebra of $\overline{\mathbb{P}}$.
- To make this a topological game, let $\mathbb{P} = \mathrm{O}(X)$, the open subsets of a space X (ordered by inclusion).

Three views of regular subalgebras

- Say a suborder $\mathbb{Q}$ of $\mathbb{P}$ is a complete suborder and write $\mathbb{Q} \subset_c \mathbb{P}$ if every maximal antichain of $\mathbb{Q}$ is also a maximal antichain of $\mathbb{P}$.
- $\mathbb{Q} \subset_c \mathbb{P}$ iff every $\mathbb{P}$-generic forcing extension contains a $\mathbb{Q}$-generic forcing extension.
- Say a Boolean subalgebra $\mathbb{A} \subset \mathbb{B}$ is a regular subalgebra and write $A \subset_{\mathrm{reg}} B$ if every maximal antichain of $\mathbb{A}$ is also a maximal antichain of $\mathbb{B}$.
- $\mathbb{A} \subset_{\mathrm{reg}} \mathbb{B}$ iff every $b \in \mathbb{B}$ has a reduction to $\mathbb{A}$: some $a \in \mathbb{A} \setminus \{0\}$ such that for all $c \in \mathbb{A}$ we have $c \not\perp a \Rightarrow c \not\perp b$.
- The Stone dual of a regular subalgebra is a continuous quotient map that is semi-open: images of nonempty open sets have nonempty interior.

Solution to the game problem

Theorem

Given a finite team size τ *and a poset* $\mathbb{P}$*, the following are equivalent.*

1. I *has an uncoordinated winning strategy for the open-open game.*
2. II *has an uncoordinated winning strategy for the regular subalgebra game.*
3. *There is a club* $\mathcal{E} \subset [\mathbb{P}]^\omega$ *such that for all* $\vec{\mathbb{Q}} \in \mathcal{E}^\tau$ *we have*

$$\left\{ \bigwedge_{i<\tau} q_i \;\middle|\; \vec{q} \in \prod_i \mathbb{Q}_i \right\} \subset_c \overline{\mathbb{P}}.$$

Topological view of the solution

Corollary

Given a finite team size τ and a nonempty space X, the following are equivalent.

1. I *has an uncoordinated winning strategy for the open-open game.*
2. II *has an uncoordinated winning strategy for the regular subalgebra game.*
3. *There is a club $\mathcal{E} \subset [\mathrm{O}(X)]^\omega$ such that for all $\vec{\mathbb{Q}} \in \mathcal{E}^\tau$ we have*

$$\left\{ \bigcap_{i<\tau} \overline{U_i}^\circ \;\middle|\; U \in \prod_i \mathbb{Q}_i \right\} \subset_c \mathrm{RO}(X)$$

where $\overline{U_i}^\circ$ is the interior of the closure of U_i.

Algebraic view of the solution

Corollary

Given a finite team size τ and a Boolean algebra $\mathbb{B}$, the following are equivalent.

1. I *has an uncoordinated winning strategy for the open-open game.*
2. II *has an uncoordinated winning strategy for the regular subalgebra game.*
3. *There is a club $\mathcal{E} \subset [\mathbb{B}]^\omega$ such that $\left\langle \bigcup_{i<\tau} A_i \right\rangle \subset_{\mathrm{reg}} \mathbb{B}$ for all $\vec{A} \in \mathcal{E}^\tau$.*

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Main Theorem

Theorem

A Boolean algebra $\mathbb{B}$ is Cohen iff it is infinite and π-homogeneous and team II has an uncoordinated winning strategy for the regular subalgebra game for all finite team sizes τ satsifying $\aleph_\tau \leq \pi(X)$.

Corollary

Given a poset $\mathbb{P}$, the following are equivalent.

- *$\mathbb{P}$ is forcing equivalent to some Cohen forcing $\mathsf{Fn}(\kappa, 2)$.*
- *The Stone space $X = \mathsf{Ult}(\overline{\mathbb{P}})$ is infinite and π-homogenous and team I has an uncoordinated winning strategy for the open-open game for all finite team sizes τ satsifying $\aleph_\tau \leq \pi(X)$.*

Proof Outline

1. Cohen algebras are infinite and π-homogeneous.
2. (Jech) If $\mathbb{C}$ is a Cohen algebra, then there is a club $\mathcal{D} \subset [\mathbb{C}]^\omega$ such that $\mathbb{A} \in \mathcal{D} \Rightarrow \mathbb{A} \subset_{\text{reg}} \mathbb{C}$ and $\mathbb{A}, \mathbb{B} \in \mathcal{D} \Rightarrow \langle \mathbb{A} \cup \mathbb{B} \rangle \in \mathcal{E}$.
3. Main Lemma. If $\mathbb{B}$ is a Boolean algebra and $\mathcal{E} \subset [\mathbb{B}]^\omega$ is a club such that $\left\langle \bigcup_{i<\tau} S_i \right\rangle \subset_{\text{reg}} \mathbb{B}$ for all $S_i \in \mathcal{E}$ and $\aleph_\tau \leq \pi(\mathbb{B})$, then $\mathbb{B}$ is tightly regularly filtered.
4. (Shapiro) If a Boolean algebra $\mathbb{B}$ is π-homogeneous and tightly regularly filtered, then $\mathbb{B}$ is Cohen.

Some definitions:

- $\mathbb{B}$ is <u>π-homogeneous</u> if there a cardinal κ such that $\forall b \in \mathbb{B}\ \min\{|S| \mid S \text{ dense in } \{a \in \mathbb{B} \mid 0 < a \leq b\}\} = \kappa$.
- $\pi(\mathbb{B}) = \min\{|S| \mid S \text{ dense in } \{a \in \mathbb{B} \mid 0 < a \leq 1\}\}$.
- $\mathbb{B}$ is <u>tightly regularly filtered</u> if there exist $A_\alpha \in [\mathbb{B}]^\omega$ for $\alpha < \pi(\mathbb{B})$ such that $\bigcup_\alpha A_\alpha$ is dense in $\mathbb{B}$ and $\left\langle \bigcup_{\beta<\alpha} A_\beta \right\rangle \subset_{\text{reg}} \left\langle \bigcup_{\beta<\alpha+1} A_\beta \right\rangle$.

A convenient version of Davies trees

- Fix a sufficiently large regular cardinal θ.
- A long ω_1-approximation sequence is a transfinite sequence $(M_\alpha)_{\alpha<\eta}$ of countable elementary submodels of $H(\theta)$ such that $(M_\beta)_{\beta<\alpha} \in M_\alpha$.
- There is a uniformly definable finite interval partition $[0,\alpha) = \bigcup\{J_i(\alpha) \mid i < \daleth(\alpha)\}$ such that:
  - For every long ω_1-approximation sequence $(M_\alpha)_{\alpha<\eta}$, $\alpha \leq \eta$, and $i < \daleth(\alpha)$, the set

    $$M_{\alpha,i} = \bigcup\{M_\beta \mid \beta \in J_i(\alpha)\}$$

    is a directed union and, hence, an elementary submodel of $H(\theta)$.
  - If $\daleth(\alpha) \geq n \geq 2$, then $\alpha \geq \omega_n$.

The partition comes from cardinal normal form: α is the sum of a unique decreasing sequence of right multiples of distinct cardinals.

Towards proving the main lemma

Main Lemma

If

- $\mathbb{B}$ *is a Boolean algebra,*
- $\mathcal{E} \subset [\mathbb{B}]^\omega$ *is a club, and*
- $\left\langle \bigcup_{i<\tau} S_i \right\rangle \subset_{\mathrm{reg}} \mathbb{B}$ *for all* $S_i \in \mathcal{E}$ *and* $\aleph_\tau \leq \pi(\mathbb{B})$*,*

then there exist $A_\alpha \in [\mathbb{B}]^\omega$ *for* $\alpha < \pi(\mathbb{B})$ *such that*

- $\bigcup_\alpha A_\alpha$ *is dense in* $\mathbb{B}$ *and*
- $\left\langle \bigcup_{\beta<\alpha} A_\beta \right\rangle \subset_{\mathrm{reg}} \left\langle \bigcup_{\beta<\alpha+1} A_\beta \right\rangle$.

To start the proof, let $A_\alpha = \mathbb{B} \cap M_\alpha$ where $(M_\alpha)_{\alpha<\pi(\mathbb{B})}$ be a long ω_1-approximation sequence such that $\mathbb{B}, \mathcal{E} \in M_0$. Then:

- $\bigcup_\alpha A_\alpha$ is dense in $\mathbb{B}$ and
- $A_\alpha, A_\alpha \cap M_{\alpha,i} \in \mathcal{E}$ for all α and $i < \daleth(\alpha)$.

Finishing the proof

- Given $\alpha < \pi(\mathbb{B})$, our goal is $\left\langle \bigcup_{\beta<\alpha} A_\beta \right\rangle \subset_{\mathrm{reg}} \left\langle \bigcup_{\beta<\alpha+1} A_\beta \right\rangle$.
- It is enough to prove $\left\langle \bigcup_{\beta<\alpha} A_\beta \right\rangle \subset_{\mathrm{reg}} \mathbb{B}$.
- Elementarity reduces this to $\left\langle \bigcup_{\beta<\alpha} (A_\alpha \cap A_\beta) \right\rangle \subset_{\mathrm{reg}} A_\alpha$.
- So, it is enough to prove $\left\langle \bigcup_{\beta<\alpha} (A_\alpha \cap A_\beta) \right\rangle \subset_{\mathrm{reg}} \mathbb{B}$.
- $\left\langle \bigcup_{\beta<\alpha} (A_\alpha \cap A_\beta) \right\rangle = \langle \bigcup_{i<\tau} (A_\alpha \cap M_{\alpha,i}) \rangle$ where $\tau = \daleth(\alpha)$.
- So, it is enough to prove $\langle \bigcup_{i<\tau} (A_\alpha \cap M_{\alpha,i}) \rangle \subset_{\mathrm{reg}} \mathbb{B}$.
- By hypothesis, $\langle \bigcup_{i<\tau} S_i \rangle \subset_{\mathrm{reg}} \mathbb{B}$ if $\aleph_\tau \leq \pi(\mathcal{B})$ and $S_i \in \mathcal{E}$.
- Recall that $A_\alpha \cap M_{\alpha,i} \in \mathcal{E}$.
- We may assume $\pi(\mathbb{B})$ is uncountable. Therefore, $\aleph_\tau \leq \pi(\mathbb{B})$.
- Therefore, $\langle \bigcup_{i<\tau} (A_\alpha \cap M_{\alpha,i}) \rangle \subset_{\mathrm{reg}} \mathbb{B}$.

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An open problem

Conjecture: The main theorem's bound on team size is optimal.

Equivalently, I conjecture that for each $n < \omega$ there is a π-homogeneous Stone space X with weight and π-weight $\aleph_n$ such that I has uncoordinated winning strategies for the open-open game for all team sizes $\tau < n$, but not for team size n.

This is open for $n \geq 3$.

I have proved an analogous optimality result for a team-game characterization of continuous retracts of powers of 2.

References

These slides are available at http://dkmj.org.
The results come from section 3 of arXiv:1607.07944v3.

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